Consequences of Ergodic Frame Flow for Rank Rigidity in Negative Curvature

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Abstract

This paper presents a rank rigidity result for negatively curved spaces. Let M be a compact manifold with negative sectional curvature and suppose that along every geodesic in M there is a parallel vector field making curvature $-a^2$ with the geodesic direction. We prove that M has constant curvature equal to $-a^2$ if M is odd dimensional, or if M is even dimensional and has sectional curvature pinched as follows: $-\Lambda^2 < K < -\lambda^2$ where $\lambda/\Lambda > .93$. When a is extremal, i.e. $-a^2$ is the curvature minimum or maximum for the manifold, this result is analogous to rank rigidity results in various other curvature settings where higher rank implies that the space is locally symmetric. In particular, this result is the first positive result for lower rank (i.e. when $-a^2$ is minimal), and in the upper rank case gives a shorter proof of the hyperbolic rank rigidity theorem of Hamenstädt, subject to the pinching condition in even dimension. We also present a rigidity result using only an assumption on maximal Lyapunov exponents in direct analogy with work done by Connell. Our proof of the main theorem uses the ergodic theory of the frame flow developed by Brin and others - in particular the transitivity group associated to this flow.

1   Introduction

Rank rigidity was first proved in the higher Euclidean rank setting by Ballmann [1] and, using different methods, by Burns and Spatzier [8]. A manifold

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is said to have higher Euclidean rank if a parallel normal Jacobi field can be found along every geodesic. Ballmann and Burns-Spatzier proved that if an irreducible, compact, nonpositively curved manifold has higher Euclidean rank, then it is locally symmetric. Ballmann's proof works for finite volume as well and the most general version of this theorem is due to Eberlein and Heber, who prove it under only a dynamical condition on the isometry group of M's universal cover [11]. Hamenstädt showed that a compact manifold with curvature bounded above by -1 is locally symmetric if along every geodesic there is a Jacobi field making curvature -1 with the geodesic direction [12]. She called this situation higher hyperbolic rank. Shankar, Spatzier and Wilking extended rank rigidity into positive curvature by defining spherical rank. A manifold with curvature bounded above by 1 is said to have higher spherical rank if every geodesic has a conjugate point at π, or equivalently, a parallel vector field making curvature 1 with the geodesic direction. They proved that a complete manifold with higher spherical rank is a compact, rank one locally symmetric space [14].

These results settle many rank rigidity questions, but leave questions about other curvature settings open (see [14] for an excellent overview). In this paper we prove the following theorem, which can be applied to various settings in negative curvature.

Theorem 1. *Let M be a compact, negatively curved manifold. Suppose that along every geodesic in M there exists a parallel vector field making sectional curvature $-a^2$ with the geodesic direction. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-\Lambda^2 < K < -\lambda^2$ with $\lambda/\Lambda > .93$ then M has constant negative curvature equal to $-a^2$.*

Note that, unlike previous rank rigidity results, Theorem 1 allows for situations where the distinguished curvature $-a^2$ is not extremal. However, the cases where $-a^2$ is extremal are of particular importance and in these situations the extremality of the distinguished curvature $-a^2$ allows the hypotheses of our theorem to be weakened, as demonstrated in section 4 of this paper. The following two results are then easy corollaries of Theorem 1:

Corollary 1. *Let M be a compact manifold with sectional curvature $-1 \leq K < 0$. Suppose that along every geodesic in M there exists a Jacobi field making sectional curvature -1 with the geodesic direction. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-1 \leq K < -\lambda^2$ with $\lambda > .93$ then M is hyperbolic.*

Corollary 2. (compare with Hamenstädt [12]) *Let M be a compact manifold with sectional curvature bounded above by -1. Suppose that along every*

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geodesic in M there exists a Jacobi field making sectional curvature −1 with the geodesic direction. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-(1/.93)^2 < K \leq -1$ then M is hyperbolic.

In Corollary 1, -1 is the curvature minimum for M and we obtain a new rank rigidity result analogous to those described above. This is the first positive result for lower rank, i.e. when the distinguished curvature value is the lower curvature bound (see section 6 for more discussion). In Corollary 2, -1 is the curvature maximum for M and we obtain a shorter proof of Hamenstädt's result, under an added pinching constraint in even dimension.

In [9], Connell showed that rank rigidity results can be obtained using only a dynamical assumption on the geodesic flow, namely an assumption on the Lyapunov exponents at a full measure set of unit tangent vectors. His paper deals with the upper rank situations treated by Ballmann, Burns-Spatzier and Hamenstädt. He proves that having the minimal Lyapunov exponent allowed by the curvature restrictions attained at a full measure set of unit tangent vectors is sufficient to apply the results of Ballman and Burns-Spatzier or Hamenstädt. In the lower rank setting of this paper, this viewpoint translates into

Theorem 2. *Let M be a compact manifold with sectional curvature $-a^2 \leq K < 0$, where $a > 0$. Suppose that for a full (Liouville) measure set of unit tangent vectors v on M the maximal Lyapunov exponent at v is a, the maximum allowed by the curvature restriction. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-a^2 \leq K < -\lambda^2$ with $\lambda/a > .93$ then M is of constant curvature $-a^2$.*

The adaptation of Connell's arguments for this setting is discussed in section 5.

The proof of Theorem 1 relies on dynamical properties of the geodesic and frame flows on negatively curved manifolds. We rely heavily on Brin's work on frame flows (see [5] for a survey), and results on the ergodicity of these flows due to Brin and Gromov in odd dimension [6] and to Brin and Karcher in even dimension [7]. These results are summarized in section 2. In particular, we utilize the transitivity group H_v, defined for any vector v in the unit tangent bundle of M, which acts on $v^\perp \subset T^1 M$. Essentially, elements of H_v correspond to parallel translations around ideal polygons in M's universal cover. Brin shows that this group is the structure group for the ergodic components of the frame flow (see e.g [5] or [4]). In section 3 we show that, subject to suitable recursion properties on these ideal polygons,

H_v preserves the parallel fields that make curvature $-a^2$ with the geodesic defined by v. We then show that these recursion properties are generic and that the elements of H_v vary continuously with the choice of ideal polygon. Thus, all of H_v respects the distinguished fields. Finally, we apply results of Brin-Gromov and Brin-Karcher on the ergodicity of the 2-frame flow which imply that H_v acts transitively on $v^\perp$ and conclude that the curvature of M is constant.

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2 Notation and background

Let us begin by fixing some notation and stating the results we will need. Let M be a compact Riemannian manifold with negative sectional curvature and let $\tilde{M}$ be its universal cover. Denote by T^1M and $T^1\tilde{M}$ the unit tangent bundles to M and $\tilde{M}$, respectively. We will denote by g_t the geodesic flow on either of these spaces, and by F_t the frame flow on the Stiefel manifold St_kM, the space of ordered orthonormal k-frames on M. St_kM is a fiber bundle over T^1M with the group $SO(n-1)$ acting on the right; St_nM is a principal bundle with $SO(n-1)$ as structure group. There are standard measures on these spaces, namely Liouville measure on T^1M and $T^1\tilde{M}$, and on St_kM the product measure of Liouville measure and the measure on the fibers inherited from the Haar measure on $SO(n-1)$. Unless otherwise specified, these will be the measures used in all that follows. Let $\gamma_v(t)$ denote the geodesic in M or $\tilde{M}$ with velocity v at time 0. We will denote by $w_v(t)$ a parallel normal vector field along $\gamma_v(t)$ making the distinguished curvature $-a^2$ with $\dot{\gamma}_v(t)$. Finally, $\langle\cdot,\cdot\rangle$ will denote the Riemannian inner product, $R(\cdot,\cdot)\cdot$ will denote the curvature tensor and $K(\cdot,\cdot)$ will denote sectional curvature.

The geodesic flow gives rise to stable and unstable foliations W_g^s and W_g^u of T^1M or $T^1\tilde{M}$. These foliations are absolutely continuous, and the geodesic flow for such an M is ergodic (proved by Anosov, see Brin's appendix to [2]). The frame flow also gives rise to stable and unstable foliations W_F^s and W_F^u as shown by Brin [3]. These foliations allow Brin to define the *transitivity group* in the following way. If v and v' are on the same leaf of W_g^s then for every n-frame α above v there is a unique n-frame α' above v' such that α and α' belong to the same leaf of W_F^s. In particular, the distance between

$F_t(\alpha)$ and $F_t(\alpha')$ approaches 0 as $t \to \infty$ (this is how one determines that α' belongs to the leaf $W_F^s(\alpha)$). Let $p(v, v')$ be the map from the fiber of $St_n M$ over v to the fiber over v' that takes each α to the corresponding α' over v'. Note $p(v, v')$ corresponds to a unique isometry between $v^\perp$ and $v'^\perp$. Once defined on n-frames, $p(v, v')$ acts on all k-frames; the action on 2-frames will be what we use in this paper and we will abuse notation by using $p(v, v')$ to denote this restricted action. One can think of $p(v, v')(\alpha)$ as the result of parallel transporting α along $\gamma_v(t)$ out 'to the boundary at infinity of $\tilde{M}$' and then back to v' along $\gamma_{v'}(t)$. If v' and v belong to the same leaf of W_g^u there is similarly an isometry corresponding to parallel translation to the boundary at infinity along γ_{-v} and back along $\gamma_{-v'}$. This defines the unstable leaves for the frame flow foliation and we will also denote this isometry by $p(v, v')$. Brin (see [5] Defn. 4.4) then defines the transitivity group at v as follows:

Definition 2.1. Given any sequence $s = \{v_0, v_1, \ldots, v_k\}$ with $v_0 = v_k = v$ such that each pair $\{v_i, v_{i+1}\}$ lies on the same leaf of W_g^s or W_g^u we have an isomorphism of $v^\perp$ given by

$$I(s) = \prod_{i=0}^{k-1} p(v_i, v_{i+1}).$$

The closure of the set of all such isometries is denoted by H_v and is called the transitivity group.

The idea of the transitivity group is that it is generated by isometries coming from parallel translation around ideal polygons in $\tilde{M}$ with an even number of sides, such as the one shown in figure 1. Note that here only 'equilateral' polygons are allowed; in figure 1 only rectangles for which $W_g^u(v_1)$ is tangent to $W_g^s(v_3)$ are permitted. We will later find it useful to allow general ideal polygons.

The definition of this group arises in Brin's analysis of the ergodic components of the frame flow. He shows in [4] that the ergodic components are subbundles of $St_k M$ with structure group a closed subgroup of $SO(n-1)$ (see also [5] section 5 for an overview). In addition, his proof demonstrates that the structure group for the ergodic component is the transitivity group (see [5] Remark 2 or [4] Proposition 2). This explicit geometric description of the ergodic components is the central tool used in our proof.

We use two results on the ergodicity of the 2-frame flow in our proof.

Theorem 2.2. (Brin-Gromov [6] Proposition 4.3) *If M has negative sectional curvature and odd dimension then the 2-frame flow is ergodic.*
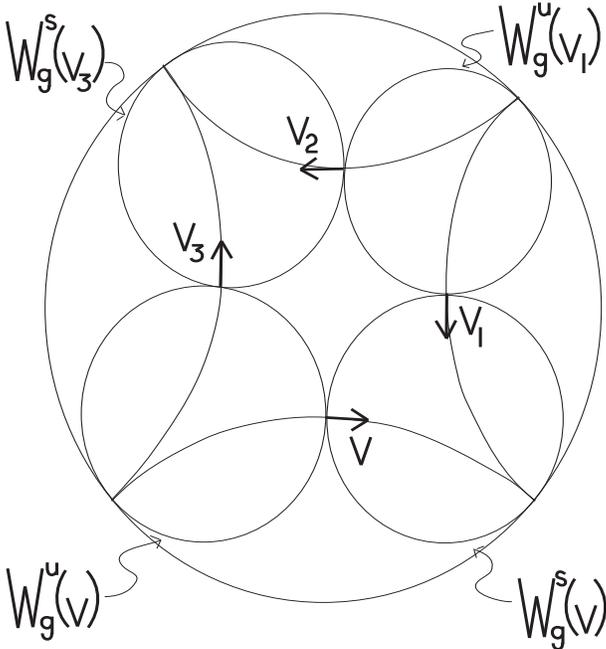
Figure 1: 'equilateral' ideal rectangle

Theorem 2.3. (Brin-Karcher [7]) *If M has sectional curvature satisfying $-\Lambda^2 < K < -\lambda^2$ with $\lambda/\Lambda > .93$ then the 2-frame flow is ergodic.*

Theorem 2.3 is not directly stated as above in [7], rather it follows from remarks made in section 2 of that paper together with Proposition 2.9 and the extensive estimates carried out in the later sections. Note that since the 2-frame flow preserves the parallel fields making curvature $-a^2$, ergodicity of this flow alone seems to indicate that the manifold has constant curvature. However, since the subset of $St_2 M$ given by these distinguished fields may, a priori, have zero measure, the result does not follow directly from ergodicity. Instead, we must use the precise description of the ergodic components given by the transitivity group.

3 The transitivity group and distinguished vector fields

As noted in the Introduction, the description of the ergodic components in terms of the transitivity group is crucial. In this section we investigate how

the distinguished vector fields $w_v(t)$ along $\gamma_v(t)$ behave under the action of the transitivity group and use the results to prove Theorem 1.

To obtain results we will need to assume some dynamical properties of the ideal polygon that produces a given element of H_v. For example, consider the ideal rectangle defined by v, v_1, v_2 and v_3 as pictured in figure 1. Note that the distinguished vector field $w_v(t)$ which makes constant curvature $-a^2$ with $\gamma_v(t)$ corresponds uniquely to a parallel normal vector field $P(t)$ along $\gamma_{v_1}(t)$, such that v_1 and $P(0)$ make a 2-frame in the leaf of the stable foliation containing the 2-frame $\{v, w_v(0)\}$, that is, $\{v_1, P(0)\} = p(v, v_1)\{v, w_v(0)\}$. By continuity of the sectional curvature, $K(P(t), \dot{\gamma}_{v_1}(t)) \to -a^2$ as $t \to \infty$.

Lemma 3.1. *Suppose $\gamma(t)$ is a recurrent geodesic with a parallel normal field $P(t)$ along it such that $K(P(t), \dot{\gamma}(t)) \to -a^2$ as $t \to \infty$. Then $K(P(t), \dot{\gamma}(t)) \equiv -a^2$ for all t.*

Proof. Since $\gamma(t)$ is recurrent we can take an increasing sequence $\{t_k\}$ tending to infinity such that $\dot{\gamma}(t_k)$ approaches $\dot{\gamma}(0)$. Since the parallel field $P(t)$ has constant norm and the set of vectors in $\dot{\gamma}_v(t)^{\perp}$ with this norm is compact, we can, by passing to a subsequence, assume that $P(t_k)$ has a limit $G(0)$. Extend $G(0)$ to a parallel vector field $G(t)$ along $\gamma(t)$.

By construction, $K(G(0), \dot{\gamma}(0)) = \lim_{k \to \infty} K(P(t_k), \dot{\gamma}(t_k)) = -a^2$. In addition, for any real number T, the recurrence $\dot{\gamma}(t_k) \to \dot{\gamma}(0)$ implies recurrence $\dot{\gamma}(t_k + T) \to \dot{\gamma}(T)$. By continuity of the frame flow, we get that $P(t_k + T) \to G(T)$ for the vector field G defined above. Thus $G(t)$ makes curvature $-a^2$ with $\dot{\gamma}(t)$ for any time t.

We can repeat the same argument as above, letting $G(t)$ recur along the same sequence of times to produce $G_1(t)$, and likewise $G_i(t)$ recur to produce $G_{i+1}(t)$, forming a sequence of fields all making curvature identically $-a^2$ with the geodesic direction. Now, observe that $G(0) = P(0) \cdot g$ for some $g \in SO(n-1)$. Note here that g is not well defined by looking at P and G alone, but will be well defined if we consider n-frame orbits with second vector P recurring to n-frames with second vector $G(0)$; this is the g we utilize. By construction and the fact that the $SO(n-1)$ action commutes with parallel translation, $G_i(0) = P(0) \cdot g^{i+1}$. $SO(n-1)$ is compact, so the $\{g^i\}$ have convergent subsequences. In addition, since the terms of this sequence are all iterates of a single element, we can, by adjusting terms of such a subsequence by suitable negative powers of g, have the subsequence converge to the identity. Choose a subsequence $\{i_j\}$ such that $g^{i_j+1} \to id$ as $j \to \infty$. These $G_{i_j}(t)$ approach our original field $P(t)$ showing that P makes constant curvature $-a^2$ with $\dot{\gamma}$ as well. $\square$

Consider the situation depicted in figure 1. Lemma 3.1 shows that, when

γ_{v_1} is recurrent in forward time, the map $p(v, v_1)$ preserves the distinguished vector fields in the sense that it sends a vector from one such field, $w_v(0)$, to a vector from another such field along γ_{v_1}. Thus, if in figure 1 we have that γ_{v_1} and γ_{v_3} are recurrent in positive time and γ_v and γ_{v_2} are recurrent in negative time, then the element of H_v given by parallel translation around this ideal rectangle will map $w_v(0)$ to another element of $v^{\perp}$ which is in a parallel field along γ_v making curvature $-a^2$. If these sort of recurrence properties held for all 'equilateral' ideal polygons based at v we would have that the transitivity group preserves the distinguished vector fields. We cannot assure that these recurrence properties are always present, but ergodicity of the geodesic flow on M indicates that they will be present almost all the time. We now work out the details of this.

First, the ergodicity of the geodesic flow implies that there is a full measure set of vectors v in T^1M which have dense forward and backward orbits under the geodesic flow. Choosing v from this set implies that γ_{v_1} will be recurrent in positive time and γ_v will be recurrent in negative time. For ideal rectangles, this leaves only the positive time recurrence of γ_{v_3} and the negative time recurrence of γ_{v_2} lacking. It is convenient at this time to extend the definition of the transitivity group.

Consider the situation depicted in figure 2. Here the unit tangent vectors v, v_1, v_2 and v_3 describe an ideal rectangle in $T^1\tilde{M}$. Each pair $\{v, v_1\}$, $\{v_1, v_2\}$, $\{v_3, v\}$ lies on a leaf of W_g^s or W_g^u and we let $T \in \mathbb{R}$ be the time such that $g_T(v_2) \in W_g^s(v_3)$ or $W_g^u(v_3)$. In order to make a true ideal rectangle we require that the leaves connecting these pairs alternate between stable and unstable. Note that in figure 2, the leaf containing the first pair, $\{v, v_1\}$ is a stable leaf, but we can similarly start with an unstable leaf. We now make the following definition:

Definition 3.2. Let v, v_1, v_2, v_3 be vectors in $T^1\tilde{M}$ describing an ideal rectangle as indicated above. Let $\tilde{F}_T$ be the restriction of the time T frame flow map to the frames based at v_2. Note that the choice of v_1 and v_3 uniquely determines this rectangle and define

$$h(v_1, v_3) = p(v_3, v) \circ p(g_T(v_2), v_3) \circ \tilde{F}_T \circ p(v_1, v_2) \circ p(v, v_1).$$

Let $\hat{H}_v$ be the closure of the group generated by all such $h(v_1, v_3)$.

Note that $\hat{H}_v$ allows parallel translations along *all* ideal rectangles based at v. Furthermore, it is easy to see that a parallel translation around any 'equilateral' ideal polygon as in the definition of H_v can be broken up into a series of translations around the general ideal rectangles allowed in the definition of $\hat{H}_v$. Thus, $\hat{H}_v \supseteq H_v$. However, $\hat{H}_v$ preserves ergodic components, as frame flow certainly preserves ergodic components. As the group
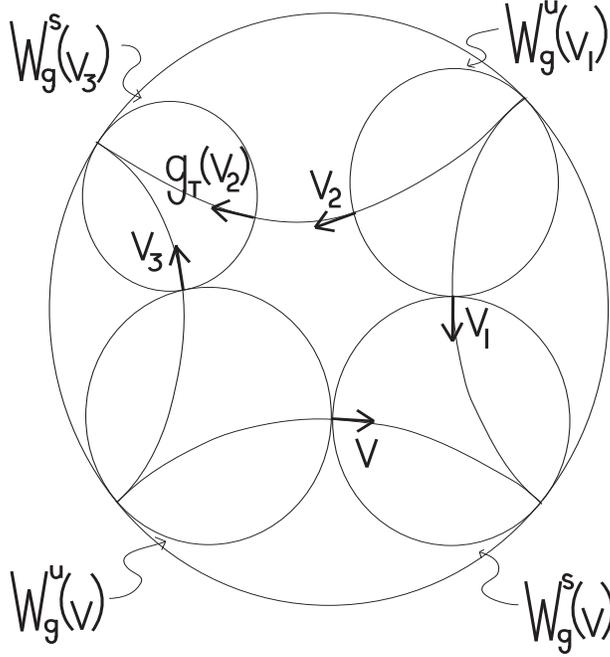
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Figure 2: general ideal rectangle

H_v is completely determined by the ergodic component of n-frames and $\hat{H}_v$ produces this same ergodic component, $\hat{H}_v \subseteq H_v$. Therefore, $\hat{H}_v = H_v$.

Despite the equality of these groups, for our purposes there is a benefit to allowing this seemingly looser definition. Each $(v_1, v_3) \in W_g^s(v) \times W_g^u(v)$ defines a rectangle used in $\hat{H}_v$. This is in opposition to the case for H_v, where only a set of measure zero define allowed rectangles. The advantage of this is the following. Since M is negatively curved, the geodesic flow is ergodic and the unstable foliation W_g^u is absolutely continuous. Thus, there is a full measure set of $v_1 \in T^1M$ with dense negative time orbit and it must intersect the leaf $W_g^s(v)$ in a set of full conditional measure for almost all $v \in T^1M$ (see Appendix to [2] Lemma 5.4). Picking v_1 from this set will ensure the needed negative time recurrence of γ_{v_2} since $\gamma_{-v_2} \to \gamma_{-v_1}$. Likewise, for almost every $v \in T^1M$ a full conditional measure set of $v_3 \in W_g^u(v)$ will have the needed positive time recurrence under the geodesic flow. We conclude that we can find a full measure set of v having dense forward and backward orbits *and* (using Fubini's theorem) with a full measure set of $W_g^s(v) \times W_g^u(v)$ possessing the desired dynamical properties for $(-v_1, v_3)$. Therefore the desired recurrence properties are generic in the set of rectangles used to generate $\hat{H}_v$ for almost all v. The final fact needed to prove that the

transitivity group preserves the distinguished vector fields is the following:

Lemma 3.3. *The map* $(v_1, v_3) \mapsto h(v_1, v_3)$ *is continuous.*

Proof. This Lemma follows from the fact that the frame flow admits a continuous foliation, which was proved by Brin [3].

First, note that $p(v, v_1)$ and $p(v_3, v)$ are defined by leaves of the foliations for the frame flow. The continuous dependence of these maps on (v_1, v_3) follows precisely from the continuity of the leaves.

Second, as (v_1, v_3) varies, the leaves $W_g^u(v_1)$ and $W_g^s(v_3)$ and the geodesic connecting $\gamma_{v_1}(-\infty)$ to $\gamma_{v_3}(\infty)$ all vary continuously. Thus v_2, T and $g_T(v_2)$ will vary continuously. Along with the argument of the previous paragraph, all this implies that the maps $p(v_1, v_2)$ and $p(g_T(v_2), v_3)$ depend continuously on (v_1, v_3). Also, $\tilde{F}_T$ will depend continuously on (v_1, v_3) as the frame flow is continuous.

Since $h(v_1, v_3)$ is the composition of these maps, the Lemma is proved. $\square$

Now we can prove the following result:

Proposition 3.4. *For almost all $v \in T^1M$, if $w_v(t)$ is a parallel field along $\gamma_v(t)$ making constant curvature $-a^2$ with the geodesic direction, then for every element h in the transitivity group we have $K(h(w_v(0)), v) = -a^2$.*

Proof. As discussed above, for almost all $v \in T^1M$ a full measure set of the $(v_1, v_3) \in W_g^s(v) \times W_g^u(v)$ give rectangles with the recurrence properties necessary for $h(v_1, v_3) \in \hat{H}(v)$ to map $w_v(0)$ to another distinguished vector field along γ_v. In particular, this set of 'nice' (v_1, v_3) is dense in $W_g^s(v) \times W_g^u(v)$. Since $h(v_1, v_3)$ depends continuously on (v_1, v_3) and for a dense set of (v_1, v_3), it preserves the distinguished fields, we have that all $h(v_1, v_3)$ preserve the distinguished fields. Since $\hat{H}_v = H_v$ is generated by these elements, the transitivity group preserves the curvature $-a^2$ as desired. $\square$

This result gives us the desired relationship between the transitivity group and the distinguished vector fields. We can now apply the results of Brin-Gromov and Brin-Karcher and prove Theorem 1 easily.

Theorem 1. *Let M be a compact, negatively curved manifold. Suppose that along every geodesic in M there exists a parallel vector field making sectional curvature $-a^2$ with the geodesic direction. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-\Lambda^2 < K < -\lambda^2$ with $\lambda/\Lambda > .93$ then M has constant negative curvature equal to $-a^2$.*

Proof. We showed in Proposition 3.4 that for almost all $v \in T^1M$ the sectional curvature $K(h(w_v(0)), v) = -a^2$ for all h in the transitivity group. In the setting of the theorem, the results of Brin-Gromov and Brin-Karcher tell us that the 2-frame flow is ergodic. In particluar, since the transitivity group gives the ergodic component for this flow, the transitivity group must act transitively on $v^\perp \subset T^1M$. Thus, $K(\cdot, v)$ is identically $-a^2$. Since this holds for almost all v it holds for all v by continuity of $K(\cdot, \cdot)$, and the theorem is proved. □

4   Parallel fields and Jacobi fields

In [14] a distinction is made between 'weak' and 'strong' rank. The existence of *parallel* fields making extremal curvature is called strong rank; the existence only of *Jacobi* fields making extremal curvature is called weak rank. A parallel field can be scaled (by a solution to the real variable version of the Jacobi equation where the standard derivative replaces the covariant derivative) to produce a Jacobi field. Thus, a proof under the less stringent hypothesis of weak rank implies a proof for strong rank. Hamenstädt's is the sole result prior to this paper for weak rank. She states her main theorem for parallel fields only, but she shows in Lemma 2.1 that in negative curvature a Jacobi field making maximal curvature is a parallel field scaled by a function [12]. Essentially, she shows that Jacobi fields making maximal curvature grow at precisely the rate one finds for the constant curvature case. Connell accomplishes the same in [9] Lemma 2.3. This, together with some of the arguments below, shows that these Jacobi fields are in fact parallel fields scaled by an appropriate function. Therefore, Corollary 2 is a weak rank result, needing only the Jacobi field hypothesis.

In this section we show that Jacobi fields making *minimal* curvature with the geodesic direction are also scaled parallel fields. This will justify the phrasing of Corollary 1 as a weak rank result.

First, note that we need only consider non-vanishing Jacobi fields; hence it will be enough to prove that stable and unstable Jacobi fields are scaled parallel fields. Stable Jacobi fields are those which have norm approaching zero as $t \to \infty$; unstable Jacobi fields have the same property in the negative time direction. Suppose $J(t)$ is a stable Jacobi field along the geodesic $\gamma(t)$ making curvature $-a^2$ with the geodesic (take $a > 0$ now), where $-a^2$ is the curvature minimum for the manifold (the modifications of what follows for unstable Jacobi fields are straightforward). The Rauch Comparison Theorem (see [10] Chapt 10, Theorem 2.3) can be used to show that

$$|J(t)| \geq |J(0)|e^{-at}. \tag{1}$$

We would like to show that equality is achieved in (1). Write $J(t) = j(t)U(t)$ where $j(t) = |J(t)|$ and $U(t)$ is a unit vector field. Then the Jacobi equation for J reads:

$$j''U + 2j'U' + jU'' + jR(\dot{\gamma}, U)\dot{\gamma} = 0 \tag{2}$$

where j' denotes the standard derivative and U' denotes covariant derivative. Taking the inner product of (2) with U and noting that $\langle U'', U \rangle = -\langle U', U' \rangle$ we obtain

$$j'' - j(\langle U', U' \rangle + a^2) = 0. \tag{3}$$

We now know the following about the magnitude of J: $j \geq 0$ by definition, $\lim_{t \to \infty} j(t) = 0$ since J is a stable Jacobi field, and $j'' \geq a^2 j$ by (3). These allow the following conclusion; its proof was shown to the author by Jeffrey Rauch:

Lemma 4.1. *Let j be a non-negative, real valued function satifsying $j'' \geq a^2 j$ and $\lim_{t \to \infty} j(t) = 0$. Then $j(t) \leq j(0)e^{-at}$.*

Proof. We have that $a^2 j - j'' \leq 0$. On the interval $0 \leq t \leq R$ for $R \gg 1$ define g_R by $g_R(0) = j(0)$, $g_R(R) = j(R)$ and $a^2 g_R - g_R'' = 0$. Note that as $R \to \infty$, $g_R \to j(0)e^{-at}$. We claim that $j \leq g_R$; the Lemma follows in the limit.

This claim is essentially the maximum principle. First, $j \leq g_R$ holds at 0 and R. Now suppose $j - g_R$ has a positive maximum at $c \in (0, R)$. Then $(j'' - g_R'')(c) \leq 0$. However, we know $a^2(j - g_R) - (j'' - g_R'') \leq 0$, so a positive value of $j - g_R$ at c together with a negative value of $j'' - g_R''$ yields a contradiction. Therefore $j \leq g_R$ holds on all of $[0, R]$ as desired. $\qquad\square$

This Lemma, together with equation (1), tells us that $|J(t)| = |J(0)|e^{-at}$. Examining equation (3) we see that having the growth rate e^{-at}, as in the constant curvature $-a^2$ case, implies that $U' = 0$, that is, J is a scaled parallel field, as desired.

5   The dynamical perspective

In this section we discuss how the results of Connell in [9] can be adapted to prove Theorem 2 as a simple consequence of Corollary 1. The necessary changes are for the most part cosmetic; the discussion here is included for

completeness, but the author does not claim to have added anything of substance to Connell's work. The notation below that has not already been assigned follows Connell's for ease of reference.

Recall that Lyapunov exponents are a tool for measuring long-term asymptotic growth rates in dynamical systems (see Katok and Mendoza's Supplement to [13] section S.2 for an exposition). In the setting of the geodesic flow they can be defined as follows. Let $v \in T^1M$ and $u \in v^\perp$. Let $J_u(t)$ be the unstable Jacobi field along γ_v with initial condition $J_u(0) = u$. Then, the *positive Lyapunov exponent at v in the u-direction* is

$$\lambda_v^+(u) = \limsup_{t\to\infty} \frac{1}{t} log|J_u(t)|.$$

Define

$$\lambda_v^+ = \max_{u\in v^\perp} \lambda_v^+(u).$$

This is the maximal Lyapunov exponent at v; the curvature bound $-a^2 \leq K$ (again, take $a > 0$) implies that $\lambda_v^+ \leq a$. Let

$$\Omega = \{v \in T^1M : \lambda_v^+ = a\}.$$

We can now rephrase Theorem 2 more succinctly.

Theorem 2. *Let M be a compact manifold with sectional curvature $-a^2 \leq K < 0$. Suppose that Ω has full measure with respect to a geodesic flow-invariant measure μ with full support. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-a^2 \leq K < -\lambda^2$ with $\lambda/a > .93$ then M is of constant curvature $-a^2$.*

Connell shows in the upper rank case that the dynamical assumption implies the geometric one, that is, that the manifold in fact has higher rank, allowing the application of an appropriate rank rigidity theorem. He first shows ([9] Proposition 2.4) that along a closed geodesic $\lambda_v^+ = a$ implies the existence of an unstable Jacobi field making curvature $-a^2$ with the geodesic direction. Essentially, if the Jacobi field giving rise to the Lyapunov exponent does not have this curvature, it will continually see non-extremal curvature a positive fraction of the time as it moves around the closed geodesic. This contradicts the supposed value of the Lyapunov exponent. The lower curvature bound version of the argument is exactly the same as that presented by Connell, with the proper inequalities reversed; also note that the work in section 4 of this paper gives the results analogous to Connell's Lemma 2.3 necessary for the argument.

It is clear that if a dense set of geodesics have the distinguished Jacobi fields, then all geodesics will. Since the velocity vectors for closed geodesics are dense in T^1M, Connell finishes his proof in section 3 of [9] by showing that these vectors are all in Ω and using the argument of the previous paragraph. Adapted to the setting of Theorem 2 the argument runs as follows. If $w \in T^1M$ is tangent to a closed geodesic and $\lambda_w^+ < a$ the previous paragraph implies that any unstable Jacobi field along γ_w must make curvature strictly greater than $-a^2$ for a positive amount of time. By continuity, this will also be true of any unstable Jacobi field along a geodesic γ_v in a sufficiently small neighborhood of γ_w (in the Sasaki metric on T^1M). The ergodic theorem implies that for a full measure set of $v \in T^1M$, γ_v will spend a positive fraction of its life in this small neighborhood of the periodic geodesic γ_w; the positivity follows from the fact that μ has full support. The intersection of this full measure set with the full measure set Ω thus contains vectors v which have $\lambda_v^+ = a$ but spend a positive fraction of their life so close to γ_w that no Jacobi fields along them can make the minimal curvature $-a^2$ with the geodesic direction during this fraction of the time. In fact, since γ_w is compact, so is the closure of this small neighborhood and therefore the curvature between these Jacobi fields and the geodesics, when in this neighborhood, can be bounded away from $-a^2$, i.e. $K(J_u, \dot\gamma_v) > c > -a^2$ for a fixed c. Having this curvature bound a positive fraction of the time contradicts $\lambda_v^+ = a$; therefore all closed geodesics must lie in Ω and the argument is complete.

Again, this version of the argument, relevant for the lower curvature bound situation, is the same as that presented by Connell with the proper inequalities reversed. Thus, the dynamical assumption implies the geometric assumption of Corollary 1 and Theorem 2 follows. Note that for these arguments the extremality of the distinguished curvature is essential; a result that parallels Theorem 1 in allowing non-extremal distinguished curvature cannot be hoped for.

6 Conclusion

We conclude with a few remarks on our results in the context of the other rank rigidity theorems. As noted above, Corollary 2 treats the case dealt with by Hamenstädt's hyperbolic rank rigidity theorem, strong upper hyperbolic rank. Unlike Hamenstädt's result, the result presented in this paper is limited by the curvature pinching condition in even dimension. However, this paper's proof is shorter, and has the advantage of telling us which symmetric space $\tilde{M}$ is. Corollary 1 is strong lower hyperbolic rank rigidity, and this result is

the first positive result for lower rank rigidity of any sort. Counterexamples to lower rank rigidity in other curvature settings are known; [14] presents an overview, together with counterexamples to weak upper and lower spherical rank rigidity. Counterexamples to lower Euclidean rank rigidity are given in [15]. When the value $-a^2$ is not extremal we have a result of a different type than previous rank rigidity results. Our results also show that spaces of constant negative curvature can not be deformed while maintaining the distinguished parallel vector fields along all geodesics, or while maintaining extremal Lyapunov exponents at a full measure set of T^1M, except by scaling the metric.

Note that in even dimension a result as extensive as our odd dimensional result cannot be hoped for. Since parallel translation preserves the complex structure on a Kähler manifold the 2-frame flow will not be ergodic (see [6] for some results on unitary frame bundles). These known counterexamples to ergodic frame flow are excluded by requiring $-1 < K < -1/4$, leading Brin to conjecture that strict 1/4-pinching implies that the frame flow is ergodic ([5] Conjecture 2.6). A positive answer to this conjecture, or any extended results for ergodicity of the 2-frame flow in negative curvature would extend the results on rank rigidity presented here correspondingly, using the same proof as presented above. One still hopes that lower hyperbolic rank rigidity (in the sense that higher rank implies the space is locally symmetric) could be true without any curvature pinching in even dimensions, perhaps even without the restriction $K < 0$, but such a result would call for a completely different method of proof from that presented here.

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