ENTROPY RIGIDITY AND HILBERT VOLUME

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Abstract. For a closed, strictly convex projective manifold that admits a hyperbolic structure, we show that the ratio of Hilbert volume to hyperbolic volume is bounded below by a constant that depends only on dimension. We also show that for such spaces, if topological entropy of the geodesic flow goes to zero, the volume must go to infinity. These results follow from adapting Besson–Courtois–Gallot's entropy rigidity result to Hilbert geometries.

1. Introduction

A *(strictly) convex real projective orbifold* is a quotient Ω/Γ, where Ω is an open, properly (strictly) convex subset of $\mathbb{RP}^n$ and $\Gamma < \mathrm{PGL}(n+1,\mathbb{R})$ is a discrete subgroup of projective transformations that preserves Ω. A subset $\Omega \subset \mathbb{RP}^n$ is *proper* if it is bounded in some affine patch; *convex* if its intersection with any projective line is connected; and *strictly convex* if, moreover, its topological boundary in an affine patch does not contain an open line segment. An orbifold is a *manifold* if Γ contains no elements of finite order.

Any properly convex set Ω admits a complete Finsler metric called the *Hilbert metric*. This is the Klein model of the hyperbolic metric when Ω is the interior of a round ball. Hence, the first examples of projective orbifolds are hyperbolic orbifolds. By Mostow rigidity, a hyperbolic structure on a closed manifold of dimension greater than or equal to 3 is unique, up to isometry. However, the dimension of the deformation space of strictly convex projective structures on some closed manifolds with $\dim \geq 3$ can be large.

It is of interest to characterize a hyperbolic structure, when it exists, among all strictly convex projective structures a manifold admits. This article addresses that question in terms of volume and entropy and derives a pair of results on the geometry and dynamics of these spaces. The Finsler structure on Ω provides a natural volume form on $Y = \Omega/\Gamma$ referred to as *Hilbert volume*. Let us assume that the manifold Y admits a hyperbolic metric g_0. Let $\mathrm{Vol}(Y, g_0)$ and $\mathrm{Vol}(Y, F_\Omega)$ denote its hyperbolic and Hilbert volumes, respectively. Our main result on Hilbert volume is given below.

Theorem 1.1. *Let Y be a closed strictly convex projective manifold which admits a hyperbolic structure. Then there exists a constant $\mathcal{D} > 0$, depending only on dimension, such that*

$$\frac{\mathrm{Vol}(Y, F_\Omega)}{\mathrm{Vol}(Y, g_0)} \geq \mathcal{D}.$$

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A consequence of the Margulis lemma [BGS85] is that there exists a positive lower bound for the volume of any hyperbolic n-manifold, for each n. This gives the following corollary.

Corollary 1.2. *Let Y be a closed strictly convex projective manifold which admits a hyperbolic structure. Then there exists a constant $\mathfrak{c} > 0$, depending only on dimension, such that*

$$\mathrm{Vol}(Y, F_\Omega) \geq \mathfrak{c}.$$

In dimension 2, Theorem 1.1 and Corollary 1.2 follow from volume bounds given in [CVV04] and [AC18].

It is important to underscore that there are strictly convex real projective manifolds which do not admit a hyperbolic metric. Coxeter group examples exist in dimension four [Ben06] and Gromov-Thurston examples exist for each dimension greater than three [Kap07]. However, our theorem holds in several contexts.

For instance, in dimensions two and three, Theorem 1.1 applies to all closed strictly convex projective manifolds by Benoist's dichotomy [Ben04, Theorem 1.1]: strict convexity of Ω is equivalent to Gromov hyperbolicity of Γ. Furthermore, there are hyperbolic manifolds with nontrivial deformation spaces of strictly convex projective structures in any dimension:

- In dimension two, there is a $16g - 16$ dimensional deformation space of strictly convex real projective structures on a closed surface of genus g [Gol90].
- There are examples of *flexible* closed hyperbolic 3-manifolds that have nontrivial projective deformations [CLT07]. In the same work, Cooper, Long and Thistlethwaite conjecture that all hyperbolic 3-manifolds are virtually flexible.
- Finally, any hyperbolic n-manifold with a totally geodesic hypersurface admits the projective bending deformation of Thurston [JM87].

Our second main result concerns the dynamics of the geodesic flow for Y.

Definition 1.3. Let g be a (Finsler or Riemannian) metric on a compact manifold Y. Let $y \in \tilde{Y}$ be any point in the universal cover of Y and $B(R, y, g)$ the radius R ball around y with respect to g. The *volume growth entropy* of g is,

$$h(g) = \lim_{R\to\infty} \frac{1}{R} \log(\mathrm{Vol}_g\, B(R, y, g)).$$

For a nonpositively curved Riemannian metric, the volume growth entropy is equal to the topological entropy for the geodesic flow [Man79]. This result can be generalized to non-Riemannian settings under some mild conditions mimicking nonpositive curvature (see [Leu06]). Verification of these conditions in the present setting can be found in [Cra09, Lemma 8.3].

We prove the following relationship between this dynamical quantity and the Hilbert volume:

Theorem 1.4. *Let $Y_t = \Omega_t/\Gamma_t$ be a family of strictly convex real projective structures on a manifold of dimension at least 2 which supports a hyperbolic metric. Then*

$$h(F_{\Omega_t}) \to 0 \;\Rightarrow\; \mathrm{Vol}(Y_t, F_{\Omega_t}) \to \infty.$$

Remark 1.5. In [Nie15, Theorem 1.4 & Corollary 1.6], Nie constructed sequences of strictly convex projective structures for which the entropy goes to zero on any closed surface of genus $g \geq 2$ and on a specific class of manifolds in dimensions 3 and 4. In these cases, volume grows without bound as entropy decreases. At about the same time, and via a different construction, Zhang produced sequences with these properties on any closed surface of genus ≥ 2 [Zha15, Corollary 3.7]. Theorem 1.4 states that this phenomenon will hold generally in any dimension with a short proof.

The $n \geq 3$ cases of Theorem 1.1 and 1.4 are consequences of an entropy rigidity theorem ($n = 2$ is treated separately). This theorem follows a line of results beginning with the celebrated work of Besson, Courtois and Gallot in [BCG95, BCG96]. They prove the following theorem using the 'barycenter method' – the technique we will also use.

Definition 1.6. The *normalized entropy functional* of a Riemannian or Finsler manifold (Y^n, g) is the scale-invariant quantity

$$\operatorname{ent}(Y, g) = h(g)^n \operatorname{Vol}(Y, g).$$

Theorem 1.7 (see Théorème Principal [BCG95])**.** *If (Y, g) is a compact, oriented, Riemannian manifold of dimension $n \geq 3$ homotopy equivalent to a negatively curved locally symmetric space (X, g_0), then*

$$\operatorname{ent}(Y, g) \geq \operatorname{ent}(X, g_0)$$

with equality if and only if (Y, g) and (X, g_0) are isometric, up to a homothety.

Remark 1.8. Theorem 1.7 has a number of important consequences, including a proof of Mostow's rigidity theorem (see [BCG95, BCG96]). The barycenter method has been employed many times, including the work of Connell and Farb on higher-rank symmetric spaces [CF03a, CF03b]. See [CF03c] for a survey.

Our paper closely follows the work of Boland and Newberger in [BN01], which adapts the Besson–Courtois–Gallot result to compact Finsler manifolds of negative flag curvature. For a C^2-Finsler metric F on a manifold, Boland and Newberger define the *eccentricity factor*, denoted by $N(F)$. See Section 3.2 for the definition, but note here that $N(F)$ is equal to 1 when F is Riemannian and is strictly greater than 1 otherwise. (The terminology 'eccentricity factor' is coined in [CF03c].) Their Finsler extension of Theorem 1.7 is as follows.

Theorem 1.9 (see Main Theorem [BN01])**.** *Let (M, F) be a compact, reversible, C^2-Finsler manifold of negative flag curvature and dimension ≥ 3 with the same homotopy type as the compact, negatively curved, locally symmetric space (X, g_0). Then*

(i) $\operatorname{ent}(X, g_0) \leq N(F) \operatorname{ent}(M, F)$.

(ii) Equality holds above if and only if (M, F) is Riemannian and homothetic to (X, g_0).

We extend this result to the Hilbert geometry setting:

Theorem 1.10. *Let (Y, F_Ω) be a compact strictly convex real projective manifold of dimension ≥ 3. Let (X, g_0) be a hyperbolic structure on the same underlying manifold. Then there is a number $N(F_\Omega) \geq 1$, such that*

$$N(F_\Omega) \operatorname{ent}(Y, F_\Omega) \geq \operatorname{ent}(X, g_0),$$

with equality if and only if (Y, F_Ω) *is isometric to* (X, g_0).

Our modifications to the work of Boland and Newberger revolve around the following point: If F_Ω is a Finsler metric defined on a strictly convex domain $\Omega \subset \mathbb{RP}^n$, one can verify (see Section 2.1) that F_Ω is C^2 if and only if $\partial\Omega$, the boundary of Ω, is C^2. If $\partial\Omega$ is C^2, then $\partial\Omega$ is in fact an ellipsoid (originally due to [Ben60, Theorem C], see also [Cra13, Section 3.2] for an expository note in English). Hence, any corresponding $Y = \Omega/\Gamma$ with the induced metric is hyperbolic. If F_Ω is not C^2, then $\partial\Omega$ is only $C^{1+\alpha}$ for some $0 < \alpha < 1$ [Ben04, Theorem 1.3]. The failure of C^2 regularity in general leads us to use the Blaschke metric, a particular Riemannian metric associated to Ω, in place of the family of reference metrics used in [BN01] at a key point in the proof. The C^2 rigidity for Hilbert geometries also allows us to reach the rigidity conclusion of Theorem 1.10 with a shorter argument.

Remark 1.11. It is conjectured in [BCG96] that Theorem 1.7 remains true in the class of Finsler metrics. This is equivalent to a restatement of Theorem 1.9 without the presence of the $N(F)$ factor. Following suit, we make the conjecture that Theorem 1.10 is valid without the $N(F_\Omega)$ term.

Outline of the paper. Section 2 provides the necessary background information on the Hilbert metric and Hilbert volume.

Section 3 contains the proof of Theorem 1.10. In Section 3.1, we construct the 'natural map' between the Hilbert geometry and its hyperbolic counterpart. In Section 3.2, we bound the Jacobian of the natural map, and deduce the inequality statement of Theorem 1.10. Finally, we prove the rigidity statement of Theorem 1.10 in Section 3.3.

Section 4 recalls basic properties of the Blaschke metric for Hilbert geometries, in particular a relationship between the Hilbert and Blaschke metrics due to Benoist and Hulin [BH13]. We then prove Theorem 1.1 and the $n \geq 3$ case of Theorem 1.4. We conclude by proving Theorem 1.4 for 2-dimensional Hilbert geometries. Since our previous results require dimension greater than two, this argument uses the well-known result of Katok on entropy rigidity for surfaces [Kat82], as well as the particularly nice behavior of the Blaschke metric in dimension 2 due again to Benoist–Hulin [BH14].

2. Hilbert Geometry

This section provides the basic definitions for Hilbert geometry. For more details the reader may consult [BK53] and [CLT15].

2.1. **Definition of a Hilbert geometry.** Let Ω be a properly convex domain of $\mathbb{RP}^n$, as defined in the Introduction. Let $\partial\Omega$ denote the topological boundary of Ω. The *Hilbert metric* d_Ω on Ω is given for $x \neq y$ by

$$d_\Omega(x, y) = \frac{1}{2}\Big|\log[p, x, y, q]\Big| = \frac{1}{2}\left|\log\left(\frac{\|y - p\| \cdot \|x - q\|}{\|y - q\| \cdot \|x - p\|}\right)\right|$$

where p, q are the intersection points, in the chosen affine patch, of $\partial\Omega$ with the projective line containing x and y, and $\|\cdot\|$ denotes Euclidean distance. (Verification that this is a metric can be found in [PT14].)

Elements of $\mathrm{PGL}(n+1, \mathbb{R})$ that preserve Ω are isometries of the Hilbert metric, since projective transformations preserve the cross-ratio. If Ω is strictly convex, these elements constitute the full group of isometries, denoted by $\mathrm{PGL}(\Omega)$ [dlH93]. A *Hilbert geometry* is a triple

$$(\Omega, d_\Omega, \mathrm{PGL}(\Omega)).$$

The Hilbert geometry where Ω is the interior of an ellipsoid is the Klein model of hyperbolic n-space, $\mathbb{H}^n$. The factor of $1/2$ in the definition of d_Ω ensures constant curvature -1.

The Hilbert metric on a properly convex domain Ω in an affine patch induces the Finsler structure

$$F_\Omega(y, v) = \frac{\|v\|}{2}\left(\frac{1}{\|y - v^-\|} + \frac{1}{\|y - v^+\|}\right)$$

where v^-, v^+ are the points of intersection of the projective line through y in the direction of v with $\partial\Omega$. Note that it is immediate from the definition that C^r-regularity of F_Ω is equivalent to C^r-regularity of $\partial\Omega$.

2.2. **Hilbert Volume.** A group $\Gamma < \mathrm{PGL}(\Omega)$ is discrete in $\mathrm{PGL}(n+1, \mathbb{R})$ if and only if Γ acts properly discontinuously on Ω [CLT15]. Therefore, for such a Γ the projective and Finsler structures of Ω descend to the quotient orbifold $Y = \Omega/\Gamma$.

The Finsler structure on Y provides a natural volume. Fix any Riemannian metric g on Y and let $B_g(1, y)$ and $B_{F_\Omega}(1, y)$ denote balls of radius 1 in T_yY with respect to g and F_Ω, respectively. Then for any $y \in Y$, the Finsler volume element is

$$dF_\Omega(y) = \frac{\mathrm{Vol}_g(B_g(1, y))}{\mathrm{Vol}_g(B_{F_\Omega}(1, y))} dg.$$

It is easy to check that the definition is independent of the choice of g. The volume of Y with respect to Finsler volume will be referred to as the *Hilbert volume* of Y.

3. Proof of Theorem 1.10

In this section we prove Theorem 1.10. As mentioned in the Introduction, the argument closely follows Boland–Newberger's adaptation of the Besson–Courtois–Gallot result to the Finsler setting. For completeness, we present the argument here, highlighting the modifications we make and referring the reader to the original papers for the details which are unchanged.

3.1. **The natural map.** Let $Y = \Omega/\Gamma$ be a compact strictly convex projective n-manifold, $n \geq 3$, which admits a hyperbolic metric. We denote by X a hyperbolic manifold homeomorphic to Y. Note that $X = \mathcal{E}/\Gamma_0$, where $\mathcal{E}$ represents an ellipsoid, $\Gamma_0 < \mathrm{PGL}(\mathcal{E})$, and $\Gamma_0 \cong \pi_1(X) \cong \pi_1(Y) \cong \Gamma$. The universal covers have natural identifications: $\tilde{Y} = \Omega$ and $\tilde{X} = \mathcal{E} = \mathbb{H}^n$ is hyperbolic n-space.

3.1.1. *Busemann functions.* Let Ω be a properly convex domain equipped with Hilbert metric d_Ω. For $p, z \in \Omega$ the *Busemann function* $B^\Omega_{p,z} : \Omega \to \mathbb{R}$ is defined by

$$B^\Omega_{p,z}(q) = d_\Omega(p, z) - d_\Omega(q, z).$$

If $\partial\Omega$ is C^1, the Busemann function can be extended so that the argument z takes values in $\Omega \cup \partial\Omega$ (cf. [Ben04, Lemme 3.4]). For $\xi \in \partial\Omega$, take

$$B^\Omega_{p,\xi}(q) := \lim_{z\to\xi} B^\Omega_{p,z}(q)$$

where $z \to \xi$ along any path. It is easy to see that $B^{\Omega}_{p,\xi}$ is 1-Lipschitz with respect to d_Ω. Geometrically, $B^{\Omega}_{p,\xi}(q)$ is the signed distance between horospheres based at ξ passing through p and q. If p is fixed as a basepoint, then $B^{\Omega}_{p,\xi}(q)$ can be viewed as a family of functions mapping Ω to $\mathbb{R}$ that is parametrized by elements in $\partial\Omega$. With this notation, Busemann functions on $\tilde{X}$ will be denoted by $B^{\mathcal{E}}_{p,\xi}$.

3.1.2. *Patterson-Sullivan measures.* The *visual boundary* of a convex domain Ω is the space of all geodesic rays based at a point modulo bounded equivalence. If Ω is strictly convex with C^1-boundary, the visual boundary of Ω coincides with $\partial\Omega$.

Suppose, furthermore, that Ω admits a cocompact action by a discrete group of projective transformations. In this case we can define the *Patterson-Sullivan* density, a family of measures $\{\mu_p\}_{p\in\Omega}$ on the boundary of Ω. The defining properties of the Patterson-Sullivan density are the following:

- (quasi-Γ-invariance) $\mu_{\gamma p} = \gamma_* \mu_p$ for all $\gamma \in \Gamma$, $p \in \Omega$
- (transformation rule) $\frac{d\mu_q}{d\mu_p}(\beta) = e^{-hB^{\Omega}_{p,\beta}(q)}$

where h is the topological entropy of the Hilbert geodesic flow or, equivalently in our setting, the volume growth entropy of (Ω, F_Ω).

The construction of the Patterson-Sullivan measures originates with the work of Patterson for Fuchsian groups [Pat76] and Sullivan for convex cocompact actions on spaces of constant negative curvature [Sul79]. The concept has been extended to many settings, the most relevant being compact negatively curved manifolds and CAT(-1) metric spaces [Kai90, Rob03]. For the familiar reader, we remark that although non-Riemannian Hilbert geometries are not negatively curved in the classical sense and are not even CAT(0), one may extend the Patterson-Sullivan theory to our setting using arguments similar to those in the classical case (see in [Cra11, §4.2]; the non-strictly convex case is treated in [Bra17, §4]).

3.1.3. *The barycenter of a measure.* Fix some basepoint $o \in \tilde{X}$. For any probability measure λ on $\partial\tilde{X}$ and $x \in \tilde{X} = \mathcal{E}$, let

$$\mathcal{B}(x,\lambda) := \int_{\alpha\in\partial\tilde{X}} B^{\mathcal{E}}_{o,\alpha}(x)\,d\lambda(\alpha).$$

The Busemann functions on $\tilde{X}$ are strictly convex along geodesic segments, hence for fixed λ, $\mathcal{B}(x,\lambda)$ has a unique minimum [BCG95, Appendix A]. Denote this minimum by $\mathrm{bar}(\lambda)$; this is the *barycenter* of λ.

It is a straightforward exercise to check that the barycenter of λ is Γ-equivariant:

$$\mathrm{bar}(\gamma_*\lambda) = \gamma \cdot \mathrm{bar}(\lambda) \text{ for all } \gamma \in \Gamma.$$

3.1.4. *The natural map.* Let $f\colon \partial\tilde{Y} \to \partial\tilde{X}$ be the Γ-equivariant homeomorphism induced by the identification of fundamental groups. A natural Γ-equivariant map from $\tilde{Y}$ to $\tilde{X}$ is constructed by associating to each $y \in \tilde{Y}$ the barycenter in $\tilde{X}$ of the push-forward of the Patterson-Sullivan measure μ_y under the map f. That is, $\tilde{\Phi}\colon \tilde{Y} \to \tilde{X}$ is given by

$$\tilde{\Phi}(y) = \mathrm{bar}(f_*\mu_y).$$

The Γ-equivariance of $\tilde{\Phi}$ follows from the Γ-equivariance of f and bar, and so it descends to a map $\Phi\colon Y \to X$. This 'natural map' is at the heart of the Besson-Courtois-Gallot approach to entropy rigidity. Theorem 1.10 will be proved by bounding the Jacobian of $\tilde{\Phi}$.

Remark 3.1. Boland and Newberger assume their Finsler manifold has negative flag curvature. This ensures that $\tilde{Y}$ is diffeomorphic to $\mathbb{R}^n$. In our setting, $\tilde{Y}$ is equal to Ω, a bounded domain in projective space, by assumption.

3.2. **The Jacobian of the natural map.** Let $v(x, \alpha)$ be the hyperbolic unit tangent vector based at $x \in \tilde{X}$ with forward endpoint under the geodesic flow $\alpha \in \partial\tilde{X}$. Note that $\mathrm{bar}(\lambda) = \hat{x}$ if and only if $d_{\hat{x}}\mathcal{B}(x, \lambda) = 0$ where d denotes the gradient. Since

$$d\mathcal{B}(x, \lambda) = \int_{\alpha \in \partial\tilde{X}} dB^{\mathcal{E}}_{o,\alpha}(x) d\lambda(\alpha)$$

and, by the geometric description of the Busemann functions in hyperbolic space, $dB^{\mathcal{E}}_{o,\alpha}(x) = -v(x, \alpha)$, we see that $\mathrm{bar}(\lambda)$ is characterized implicitly by the condition

$$0 = \int_{\alpha \in \partial\tilde{X}} v(\mathrm{bar}(\lambda), \alpha) d\lambda(\alpha).$$

Therefore, $\tilde{\Phi}(y)$ satisfies

$$0 = \int_{\alpha \in \partial\tilde{X}} v(\tilde{\Phi}(y), \alpha) d(f_*\mu_y)(\alpha).$$

Changing variables by setting $\beta = f^{-1}(\alpha)$, and by the transformation rule, we have for a fixed $p \in \tilde{Y}$ and for all $y \in \tilde{Y}$,

$$0 = \int_{\beta \in \partial\tilde{Y}} v(\tilde{\Phi}(y), f(\beta)) e^{-h(F_\Omega) B^{\Omega}_{p,\beta}(y)} d\mu_p(\beta).$$

This expression allows us to verify that $\tilde{\Phi}$ is differentiable. Let

$$F(x, y) = \int_{\beta \in \partial\tilde{Y}} v(x, f(\beta)) e^{-h(F_\Omega) B^{\Omega}_{p,\beta}(y)} d\mu_p(\beta).$$

Clearly v, and therefore F, is smooth in its first variable. Since $\partial\Omega$ is $C^{1+\alpha}$, $B^{\Omega}_{p,\beta}(y)$, and therefore F, is differentiable in y (see [Ben04, §3.24]). F implicitly defines $\tilde{\Phi}$ by $F(\tilde{\Phi}(y), y) = 0$ and the Implicit Function Theorem implies that $\tilde{\Phi}$ is differentiable.

We compute the Jacobian of $\tilde{\Phi}$ as follows. Let $\langle -, - \rangle$ denote the inner product with respect to g_0. For our fixed $p \in \tilde{Y}$, all $y \in \tilde{Y}$, and all $w \in T_{\tilde{\Phi}(y)}\tilde{X}$,

$$0 = \int_{\beta \in \partial\tilde{Y}} \langle v(\tilde{\Phi}(y), f(\beta)), w \rangle e^{-h(F_\Omega) B^{\Omega}_{p,\beta}(y)} d\mu_p(\beta).$$

We want to take the differential of this expression with respect to y in an arbitrary direction $u \in T_y\tilde{Y}$. Extend w locally to a smooth vector field W on $\tilde{X}$ around $\tilde{\Phi}(y)$ so that $\nabla_{D_y\tilde{\Phi}(u)} W = 0$, where ∇ is the Levi-Civita connection for g_0. Let $V \colon T(T\tilde{X}) \to T\tilde{X}$ denote taking the vertical part with respect to ∇. Taking the differential at y of

$$0 = \int_{\beta \in \partial\tilde{Y}} \langle v(\tilde{\Phi}(y), f(\beta)), W(\tilde{\Phi}(y)) \rangle e^{-h(F_\Omega) B^{\Omega}_{p,\beta}(y)} d\mu_p(\beta)$$

in the direction $u \in T_y\tilde{Y}$ gives

$$\int_{\beta\in\partial\tilde{Y}} \langle V[Dv_{(\tilde{\Phi}(y),f(\beta))} D_y\tilde{\Phi}(u)], w\rangle d\mu_y(\beta) \\ = h(F_\Omega)\int_{\beta\in\partial\tilde{Y}} \langle v(\tilde{\Phi}(y), f(\beta)), w\rangle D_y B^{\Omega}_{p,\beta}(u) d\mu_y(\beta) \tag{3.1}$$

for all $w \in T_{\tilde{\Phi}(y)}\tilde{X}$ and all $u \in T_y\tilde{Y}$.

Let K and H be the endomorphisms on $T_{\tilde{\Phi}(y)}\tilde{X}$ defined by

$$\langle K(w'), w\rangle = \int_{\alpha\in\partial\tilde{X}} \langle V[Dv_{(\tilde{\Phi}(y),\alpha)} w'], w\rangle d(f_*\mu_y)(\alpha)$$

and

$$\langle H(w), w\rangle = \int_{\alpha\in\partial\tilde{X}} \langle v(\tilde{\Phi}(y),\alpha), w\rangle^2 d(f_*\mu_y)(\alpha).$$

The reader can verify (or see the nice exposition in [Fer96]) that H and K are symmetric, $\mathrm{tr}(H) = 1$, and (since $\tilde{X}$ is hyperbolic space) that $K = I - H$. Then from equation (3.1) and an application of Cauchy-Schwarz, we have

$$|\langle K\circ D_y\tilde{\Phi}(u), w\rangle| = h(F_\Omega)\left|\int_{\beta\in\partial\tilde{Y}} \langle v(\tilde{\Phi}(y), f(\beta)), w\rangle D_y B^{\Omega}_{p,\beta}(u) d\mu_y(\beta)\right| \\ \leq h(F_\Omega)\langle H(w), w\rangle^{\frac{1}{2}} \left(\int_{\beta\in\partial\tilde{Y}} D_y B^{\Omega}_{p,\beta}(u)^2 d\mu_y(\beta)\right)^{\frac{1}{2}}. \tag{3.2}$$

Since the goal is to bound $|\mathrm{Jac}(\Phi)|$ from above, we assume without loss of generality that $D_y\tilde{\Phi}$ has full rank. Fix a basis $\{e_i\}$ for $T_{\tilde{\Phi}(y)}\tilde{X}$ which is orthonormal with respect to the hyperbolic metric and in which the matrix for H is diagonal. Fix a Riemannian metric g_r on Y; we will specify a choice for this metric in Section 4.

Let $v_i' = (K\circ D_y\tilde{\Phi})^{-1}(e_i)$ and apply Gram-Schmidt to get a basis $\{v_i\}$ for $T_y\tilde{Y}$ which is orthogonal with respect to g_r. Then

$$K\circ D_y\tilde{\Phi}\colon T_y\tilde{Y} \to T_{\tilde{\Phi}(y)}\tilde{X}$$

is upper-triangular with respect to this basis.

Let $P\{v_i\}$ be the parallelepiped spanned by $\{v_i\}$. We have

$$\mathrm{Jac}(\tilde{\Phi})(y) = \frac{\det D_y\tilde{\Phi}}{\mathrm{Vol}_{F_\Omega}(P\{v_i\})}$$

where the determinant is computed for the matrix with respect to the bases $\{e_i\}$ and $\{v_i\}$. Since $\{v_i\}$ is orthonormal for g_r, the g_r-volume of $P\{v_i\}$ is 1. Hence, by the definition of dF_Ω,

$$\mathrm{Vol}_{F_\Omega}(P\{v_i\}) = \frac{\mathrm{Vol}_{g_r}(B_{g_r}(1,y))}{\mathrm{Vol}_{g_r}(B_{F_\Omega}(1,y))} =: \rho(y, g_r).$$

Thus we have that

$$|\mathrm{Jac}(\tilde{\Phi})(y)| = \frac{|\det D_y\tilde{\Phi}|}{\rho(y,g_r)}.$$

Since

$$|\det(K\circ D_y\tilde{\Phi})| = |\,\mathrm{Jac}(\tilde{\Phi})(y)|\cdot\rho(y,g_r)\cdot|\det K|,$$

we can use equation (3.2), the fact that $K \circ D_y\tilde{\Phi}$ is upper-triangular, and the fact that H is diagonal to compute as follows:

$$
\begin{aligned}
|\operatorname{Jac}(\tilde{\Phi})(y)| \cdot \rho(y, g_r) \cdot |\det K| &= \prod_{i=1}^{n} |\langle K \circ D_y\tilde{\Phi}(v_i), e_i\rangle| \\
&\leq h(F_\Omega)^n \prod_{i=1}^{n} \langle H(e_i), e_i\rangle^{\frac{1}{2}} \prod_{i=1}^{n} \Big(\int_{\beta\in\partial\tilde{Y}} D_y B^{\Omega}_{p,\beta}(v_i)^2 d\mu_y(\beta)\Big)^{\frac{1}{2}} \\
&= h(F_\Omega)^n \det(H)^{\frac{1}{2}} \Big[\big(\prod_{i=1}^{n} \int_{\beta\in\partial\tilde{Y}} D_y B^{\Omega}_{p,\beta}(v_i)^2 d\mu_y(\beta)\big)^{\frac{1}{n}} \Big]^{\frac{n}{2}} \\
&\leq h(F_\Omega)^n \det(H)^{\frac{1}{2}} \Big[\frac{1}{n} \sum_{i=1}^{n} \int_{\beta\in\partial\tilde{Y}} D_y B^{\Omega}_{p,\beta}(v_i)^2 d\mu_y(\beta) \Big]^{\frac{n}{2}}.
\end{aligned}
\tag{3.3}
$$

At the last step we use the fact that the arithmetic mean bounds the geometric mean from above. Since the $\{v_i\}$ are orthonormal for g_r,

$$
\begin{aligned}
\sum_{i=1}^{n} D_y B^{\Omega}_{p,\beta}(v_i)^2 = \|D_y B^{\Omega}_{p,\beta}\|^2_{g_r} &= \max_{v\in S_{g_r}(1,y)} D_y B^{\Omega}_{p,\beta}(v)^2 \\
&= \max_{v\in S_{g_r}(1,y)} F_\Omega(v)^2 D_y B^{\Omega}_{p,\beta}(\hat{v})^2
\end{aligned}
\tag{3.4}
$$

for $\hat{v} = \frac{v}{F_\Omega(v)}$. As $B^{\Omega}_{p,\beta}$ is 1-Lipschitz with respect to d_Ω and $\hat{v}$ is a unit vector for F_Ω, $D_y B^{\Omega}_{p,\beta}(\hat{v})^2 \leq 1$. Then combining (3.4) with equation (3.3) proves

$$
|\operatorname{Jac}(\tilde{\Phi})(y)| \leq \frac{h(F_\Omega)^n}{n^{\frac{n}{2}}} \frac{(\det H)^{\frac{1}{2}}}{|\det K|} \frac{\max_{v\in S_{g_r}(1,y)} F_\Omega(v)^n}{\rho(y, g_r)}. \tag{3.5}
$$

If F_Ω were Riemannian, we could set $g_r = F_\Omega$ and the third factor in the product above would be equal to 1. As F_Ω may not be Riemannian, Boland–Newberger make the following definition:

Definition 3.2 (compare with [BN01, p. 3])**.** Let (Y, F) be any Finsler manifold, and let g_r be a Riemannian metric on Y. We define the *eccentricity factor of F* as

$$
N(F) := \max_{y\in Y} \max_{v\in S_{g_r}(1,y)} \frac{F(v)^n \operatorname{Vol}_{g_r}(B_F(1,y))}{\operatorname{Vol}_{g_r}(B_{g_r}(1,y))}.
$$

Note that $\operatorname{Vol}_{g_r}(B_{g_r}(1,y))$ is a constant depending only on the dimension of Y. It is also easy to check that $N(F)$ is unchanged by scaling the Riemannian metric g_r. The following lemma is a straight-forward exercise:

Lemma 3.3. *For any Finsler manifold (Y, F) and any g_r, $N(F) \geq 1$. Furthermore, $N(F) = 1$ if and only if for all $y \in Y$, F_Ω and the norm induced by g_r are homothetic.*

Proof. Fix any $y \in Y$ and take $v \in S_{g_r}(1,y)$ which maximizes $F(v)$. Since $F(w) \leq F(v)$ for all $w \in S_{g_r}(1,y)$, $B_{g_r}(1,y) \subseteq B_F(F(v),y)$. Therefore,

$$
\frac{F(v)^n \operatorname{Vol}_{g_r}(B_F(1,y))}{\operatorname{Vol}_{g_r}(B_{g_r}(1,y))} = \frac{\operatorname{Vol}_{g_r}(B_F(F(v),y))}{\operatorname{Vol}_{g_r}(B_{g_r}(1,y))} \geq 1.
$$

Equality holds if and only if $F(v)$ is constant on $S_{g_r}(1,y)$, i.e., if and only if F_Ω is homothetic to the norm on T_yY induced by g_r. $\square$

Returning to our bounds in equation (3.5) and using Definition 3.2, for all $y \in Y$,

$$|\operatorname{Jac}(\tilde{\Phi})(y)| \leq \frac{h(F_\Omega)^n}{n^{\frac{n}{2}}} \frac{(\det H)^{\frac{1}{2}}}{|\det K|} N(F_\Omega). \tag{3.6}$$

Remark 3.4. A careful reading of [BN01] shows that, rather than a single Riemannian metric g_r, one can run the argument above using a family of Riemannian metrics $\{g_u\}$ parametrized by F_Ω-unit tangent vectors u. (The definition of $N(F_\Omega)$ is adjusted accordingly.) We do not exploit this additional flexibility here.

Boland–Newberger use $\{g_u\}$ defined by

$$g_u(v, w) = \sum_{i,j=1}^{n} (g_u)_{ij} \quad \text{where} \quad (g_u)_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{y}_i \dot{y}_j}(y, u).$$

This 'direction-dependent' inner product on $T_y\tilde{Y}$ is a standard tool in Finsler geometry (see [BCS00, §1.2 B]), but it requires at least C^2 regularity of F, which (as noted in the Introduction) we do not have unless $(\tilde{Y}, F_\Omega)$ immediately reduces to the hyperbolic case [Ben60]. A key step in our argument (see Section 4) is finding a good replacement for $\{g_u\}$.

The following lemma is where $n \geq 3$ is required for the proof of Theorem 1.10. Its proof is an optimization exercise.

Lemma 3.5 (see Appendice B, [BCG95]). *Assume $n \geq 3$. For a symmetric, $n \times n$ matrix H with trace 1 and $K = I - H$,*

(i) $\frac{(\det H)^{\frac{1}{2}}}{|\det K|} \leq \left(\frac{\sqrt{n}}{n-1}\right)^n$ *and*

(ii) if equality holds, $H = \frac{1}{n}I$ *and therefore* $K = \frac{n-1}{n}I$.

Applying Lemma 3.5 to equation (3.6) (recalling that $h(g_0) = n - 1$ for a hyperbolic n-manifold) and then integrating the result over $y \in Y$ gives

$$\frac{\operatorname{Vol}(\tilde{X}, g_0)}{\operatorname{Vol}(\tilde{Y}, F_\Omega)} \leq \left(\frac{h(F_\Omega)}{h(g_0)}\right)^n N(F_\Omega) \tag{3.7}$$

which is equivalent to the inequality statement in Theorem 1.10.

3.3. **Rigidity.** We now turn to the rigidity part of Theorem 1.10. Suppose that equality holds in (3.7). This forces the equality case of Lemma 3.5, i.e.

$$K = \frac{h(g_0)}{n} I \quad \text{and} \quad H = \frac{1}{n} I.$$

Then equation (3.2) gives

$$\frac{h(g_0)}{n} |\langle D\tilde{\Phi}(u), w\rangle| \leq \frac{h(F_\Omega)}{n^{1/2}} \|w\|_{g_0} \Big(\int_{\beta \in \partial \tilde{Y}} D_y B^\Omega_{p,\beta}(u)^2 d\mu_y(\beta) \Big)^{\frac{1}{2}}$$

for all $u \in T_y\tilde{Y}$ and $w \in T_{\tilde{\Phi}(y)}\tilde{X}$. Solving for $|\langle D\tilde{\Phi}(u), w\rangle|$ and taking the supremum over all $w \in T^1_{\tilde{\Phi}(y)}\tilde{X}$ gives

$$\|D\tilde{\Phi}(u)\|_{g_0} \leq n^{1/2} \frac{h(F_\Omega)}{h(g_0)} \Big(\int_{\beta \in \partial \tilde{Y}} D_y B^\Omega_{p,\beta}(u)^2 d\mu_y(\beta) \Big)^{\frac{1}{2}}$$

for all $u \in T_y\tilde{Y}$.

Let $L = (D_y\tilde{\Phi})^* \circ (D_y\tilde{\Phi})$, where A^* denotes the transpose (with respect to g_r and g_0) of a linear map $A\colon T_y\tilde{Y} \to T_{\tilde{\Phi}(y)}\tilde{X}$. Fix a g_r-orthonormal basis $\{u_i\}$. We then calculate:

$$\begin{aligned}
\operatorname{tr}(L) &= \sum_{i=1}^{n} g_r(Lu_i, u_i) \\
&= \sum_{i=1}^{n} \langle D_y\tilde{\Phi}(u_i), D_y\tilde{\Phi}(u_i)\rangle \\
&\leq n\left(\frac{h(F_\Omega)}{h(g_0)}\right)^2 \sum_{i=1}^{n} \int_{\beta\in\partial\tilde{Y}} D_yB^{\Omega}_{p,\beta}(u_i)^2 d\mu_y(\beta) \\
&\leq n\left(\frac{h(F_\Omega)}{h(g_0)}\right)^2 \max_{v\in S_{g_r}(1,y)} F_\Omega(v)^2
\end{aligned} \tag{3.8}$$

where we apply the same reasoning following equation (3.4) to the last line. Equality in equation (3.5) together with $K = \frac{h(g_0)}{n}I$ and $H = \frac{1}{n}I$ implies

$$\begin{aligned}
\left(\frac{h(F_\Omega)}{h(g_0)}\right)^{2n} \frac{\max_{v\in S_{g_r}(1,y)} F_\Omega(v)^{2n}}{\rho(y,g_r)^2} &= |\operatorname{Jac}(\tilde{\Phi}(y))|^2 \\
&= \frac{\det L}{\rho(y,g_r)^2} \\
&\leq \frac{1}{\rho(y,g_r)^2}\left(\frac{\operatorname{tr}(L)}{n}\right)^n \\
&\leq \left(\frac{h(F_\Omega)}{h(g_0)}\right)^{2n} \frac{\max_{v\in S_{g_r}(1,y)} F_\Omega(v)^{2n}}{\rho(y,g_r)^2}
\end{aligned} \tag{3.9}$$

using (3.8). Equality must hold throughout, in particular when we invoke $(\det L)^{1/n} \leq \operatorname{tr}(L)/n$. Equality implies that

$$L = \left(\frac{h(F_\Omega)}{h(g_0)}\right)^2 \max_{v\in S_{g_r}(1,y)} F_\Omega(v)^2 I.$$

Recalling the definition of L, this implies that for all y,

$$D_y\tilde{\Phi}\colon (T_y\tilde{Y}, F_\Omega) \to (T_{\tilde{\Phi}(y)}\tilde{X}, g_0)$$

is an isometry composed with a homothety.

We can now conclude the proof of Theorem 1.10 with a short argument, using special rigidity properties of Hilbert geometries. That $D_y\tilde{\Phi}$ is a homothety implies that $S_{F_\Omega}(1,y)$ is an ellipsoid; in particular it is smooth. Using the definition of F_Ω, this implies that $\partial\Omega$ is smooth. Then by [Ben60] and our choice of normalization for d_Ω, $(\tilde{Y}, F_\Omega)$ is in fact hyperbolic, so by Mostow's rigidity theorem [Mos68] we conclude (Y, F_Ω) and (X, g_0) are isometric.

4. Proofs of Theorems 1.1 and 1.4

Any properly convex domain Ω in $\mathbb{RP}^n$ admits a Riemannian metric called the Blaschke metric (cf. [BH13, Section 2.1], where this metric is referred to as the affine metric). The Blaschke metric is projectively invariant and agrees with the Hilbert metric if Ω is an ellipsoid, which is the case when (Ω, F_Ω) is isometric to hyperbolic n-space [Tho17, Proposition 1.6]. Let F_Ω^H and F_Ω^B denote the Hilbert and Blaschke norms on Ω, respectively.

Theorem 4.1 ([BH13, Proposition 3.4])**.** *Given any properly convex domain Ω in $\mathbb{RP}^n$, there exists a constant $K_n \geq 1$ depending only on n such that for all $v \in T\Omega$,*

$$\frac{1}{K_n} F_\Omega^H(v) \leq F_\Omega^B(v) \leq K_n F_\Omega^H(v).$$

The eccentricity factor of Definition 3.2 for the Hilbert metric with respect to the Blaschke metric is

$$N(F_\Omega) = \max_{y \in Y} \max_{v \in S_{F_\Omega^B}(1,y)} \frac{F_\Omega^H(v)^n \operatorname{Vol}_{F_\Omega^B}(B_{F_\Omega^H}(1,y))}{\operatorname{Vol}_{F_\Omega^B}(B_{F_\Omega^B}(1,y))}.$$

It follows that

$$N(F_\Omega) \leq K_n^{2n} \tag{4.1}$$

since, by Theorem 4.1, $B_{F_\Omega^H}(1,y) \subset B_{F_\Omega^B}(K_n,y)$ and $S_{F_\Omega^B}(1,y) \subset B_{F_\Omega^H}(K_n,y)$ for all $y \in \Omega$.

Proof of Theorem 1.1. By Theorem 1.10,

$$N(F_\Omega^H) h(F_\Omega^H)^n \operatorname{Vol}(Y, F_\Omega^H) \geq h(g_0)^n \operatorname{Vol}(X, g_0)$$

with equality only when (Y, F_Ω^H) and (X, g_0) are isometric. Moreover, since (X, g_0) has constant curvature -1, $h(F_\Omega^H) \leq n-1 = h(g_0)$ [Cra09]. Thus,

$$\operatorname{Vol}(Y, F_\Omega^H) \geq \frac{1}{N(F_\Omega^H)} \left(\frac{h(g_0)}{h(F_\Omega^H)}\right)^n \operatorname{Vol}(X, g_0) \geq \frac{1}{K_n^{2n}} \operatorname{Vol}(X, g_0).$$

Theorem 1.1 follows. □

Theorem 1.4 immediately follows from equation (4.1) and Theorem 1.10 for dimensions $n \geq 3$ since $h(g_0)^n \operatorname{Vol}(X, g_0)$ is constant. We treat the $n = 2$ case separately in the next section.

4.1. **Entropy and volume in dimension 2.**

Theorem 4.2. *Let $Y_t = \Omega_t/\Gamma_t$ be a family of convex projective manifolds homeomorphic to a closed surface Σ of negative Euler characteristic. Then*

$$h(F_{\Omega_t}^H) \to 0 \;\Rightarrow\; \operatorname{Vol}(Y_t, F_{\Omega_t}^H) \to \infty.$$

The approach is comparison with the Riemannian Blascke metric d_B on Ω via Theorem 4.1, and an application of Katok's entropy-rigidity theorem [Kat82, Theorem B].

In this section, we write $\operatorname{Vol}_{d_\Omega^*}$ for the volume *on Ω* induced by the metric F_Ω^*, and we write $\operatorname{Vol}_{F_\Omega^*}$ for the volume this metric induces *on the tangent space at a point*.

Lemma 4.3. *There is a constant V_n depending only on dimension such that for any measurable set $U \subset \Omega$,*

$$\frac{1}{V_n} \operatorname{Vol}_{d_\Omega^B}(U) \leq \operatorname{Vol}_{d_\Omega^H}(U) \leq V_n \operatorname{Vol}_{d_\Omega^B}(U). \tag{4.2}$$

Moreover, we have

$$h(F_\Omega^B) \leq K_n h(F_\Omega^H) \tag{4.3}$$

where K_n is as in Theorem 4.1.

Proof. Since the Blaschke metric is Riemannian we may use it for the definition of the Finsler volume. Then

$$\mathrm{Vol}_{d^H_\Omega}(U) = \int_U \frac{\mathrm{Vol}_{F^B_\Omega}(B_{F^B_\Omega}(1,x))}{\mathrm{Vol}_{F^B_\Omega}(B_{F^H_\Omega}(1,x))} d\,\mathrm{Vol}_{F^B_\Omega}(x).$$

By Theorem 4.1 and basic properties of any volume form on $T_x\Omega$,

$$\begin{aligned}\frac{1}{K_n^n}\mathrm{Vol}_{F^B_\Omega}(B_{F^B_\Omega}(1,x)) = \mathrm{Vol}_{F^B_\Omega}(B_{F^B_\Omega}(1/K_n,x)) \leq \mathrm{Vol}_{F^B_\Omega}(B_{F^H_\Omega}(1,x))\\ \leq \mathrm{Vol}_{F^B_\Omega}(B_{F^B_\Omega}(K_n,x)) = K_n^n\,\mathrm{Vol}_{F^B_\Omega}(B_{F^B_\Omega}(1,x)).\end{aligned}$$

Equation (4.2) follows with $V_n = K_n^n$.

By Theorem 4.1 and equation (4.2),

$$\begin{aligned} h(F^B_\Omega) &= \lim_{r\to\infty}\frac{1}{r}\log \mathrm{Vol}_{d^B_\Omega}\left(B_{d^B_\Omega}(r,x)\right)\\ &\leq \lim_{r\to\infty}\frac{1}{r}\log \mathrm{Vol}_{d^B_\Omega}\left(B_{d^H_\Omega}(K_n r,x)\right)\\ &\leq K_n \lim_{r\to\infty}\frac{1}{K_n r}\log \mathrm{Vol}_{d^H_\Omega}\left(B_{d^H_\Omega}(K_n r,x)\right) = K_n h(F^H_\Omega).\end{aligned}$$

This proves equation (4.3). □

The last piece needed to prove Theorem 4.2 is another result of Benoist and Hulin:

Lemma 4.4 ([BH14, Proposition 3.3])**.** *The curvature of the Blaschke metric on a properly convex* $\Omega \subset \mathbb{R}\mathrm{P}^2$ *is bounded between* -1 *and 0.*

By [Man79], this implies that topological entropy of the Blaschke geodesic flow of the quotient manifold is equal to the volume growth entropy of the Blaschke metric.

Proof of Theorem 4.2. Suppose $h(F^H_{\Omega_t}) \to 0$. By equation (4.3), $h(F^B_{\Omega_t}) \to 0$ as well. By Theorem B in [Kat82],

$$(h(F^B_{\Omega_t}))^2\,\mathrm{Vol}_{d^B_\Omega}(Y_t) \geq -2\pi E(\Sigma)$$

where $E(\Sigma)$ is the Euler characteristic of Σ. Therefore, $\mathrm{Vol}_{d^B_\Omega}(Y_t) \to \infty$ and $\mathrm{Vol}_{d^H_\Omega}(Y_t) \to \infty$ as well by equation (4.2). □

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