QUANTITATIVE SHRINKING TARGET PROPERTIES FOR ROTATIONS AND INTERVAL EXCHANGES

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1. Introduction

Let $\alpha \in [0,1)$ and λ denote Lebesgue measure on $[0,1)$. The rotation $R_\alpha : [0,1) \to [0,1)$ by $R_\alpha(x) = x + \alpha \mod 1$ is one of the most natural and best understood dynamical systems. For example, Herman Weyl proved the following result on the asymptotic frequency with which an orbit visits a fixed ball:

Theorem. *Let $\alpha \notin \mathbb{Q}$. Then for any $\epsilon > 0$ we have*

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} \chi_{B(\frac{1}{2},\epsilon)}(R_\alpha^i x)}{N 2\epsilon} = 1$$

This paper concerns the following question: What if the ball's radius is allowed to shrink as i increases? The focus of this paper is on treating families of sequences of radii $\{r_i\}$ simultaneously and obtaining explicit conditions on α under which theorems like the above can be proved. The following is the main result of this paper for rotations:

Theorem 1. *There exists an explicit full measure diophantine condition on α so that if α satisfies this condition then for any sequence $\{r_i\}$ so that ir_i is non-increasing and $\sum_{i=1}^{\infty} r_i = \infty$ we have*

$$\lim_{N\to\infty} \frac{\sum\limits_{i=1}^{N} \chi_{B(0,r_i)}(R_\alpha^i x)}{\sum\limits_{i=1}^{N} 2r_i} = 1 \tag{1}$$

for almost every x.

If α is badly approximable (a measure zero, full Hausdorff dimension set) then we can relax the condition on the radius sequences further:

Theorem 2. *Let $y \in [0,1)$. If α is badly approximable, $\{r_i\}_{i=1}^{\infty}$ is non-increasing and $\sum r_i = \infty$, then*

$$\lim_{N\to\infty} \frac{\sum\limits_{i=1}^{N} \chi_{B(y,r_i)}(R_\alpha^i x)}{\sum\limits_{i=1}^{N} 2r_i} = 1$$

for almost every x.

We note that Kurzweil showed that the conclusion of Theorem 2 can hold at most for badly approximable α:

Theorem. *(Kurzweil [22]) For any decreasing sequence of positive real numbers $\{r_i\}_{i=1}^{\infty}$ with divergent sum there exists $\mathcal{V} \subset [0,1)$, a full measure set of α, such that for all $\alpha \in \mathcal{V}$ we have*

$$\lambda\Big(\bigcap_{n=1}^{\infty} \bigcup_{i}^{\infty} B(R_\alpha^{-i}(x), r_i)\Big) = 1$$

for every x.

On the other hand,

$$\lambda\Big(\bigcap_{n=1}^{\infty} \bigcup_{i}^{\infty} B(R_\alpha^{-i}(x), r_i)\Big) = 1$$

for every x and every decreasing sequence of positive real numbers $\{r_i\}_{i=1}^{\infty}$ with divergent sum iff α is badly approximable.

Let us make a few remarks to make the statements of Theorems 1 and 2 precise. We call a sequence $\{r_i\}$ where ir_i is non-increasing and $\sum r_i = \infty$ a *Khinchin sequence*. Let $[a_1, ...]$ be the continued fraction expansion of α. The number α is *badly approximable* if $\limsup\limits_{n\to\infty} a_n < \infty$. The diophantine condition in Theorem 1 is as follows:

- $a_n < n^{\frac{4}{3}}$ for all but finitely many n and
- $\lim\limits_{C\to\infty} \limsup\limits_{N\to\infty} \frac{1}{N}(\sum\limits_{i=1}^{N} \log a_i - \sum\limits_{a_i<C}^{N} \log a_i) = 0$.

We will prove our results not just for rotations, but also for interval exchange transformations (IET's; Definition 2) satisfying similar diophantine assumptions. The statement of this more general theorem (Theorem 4) requires a few technical definitions and so is delayed until Section 2. We mention D. Kim and S. Marmi [19], S. Galatolo [13], L. Marchese [24], M. Boshernitzan and J. Chaika [6], M. Marmi, S. Moussa and J-C Yoccoz [25] where a variety of diophantine results for interval exchanges and rotations are proven.

A key tool in extending our work to IET's is a quantitative version of Boshernitzan's criterion for unique ergodicity which may be of independent interest (see Section 4 for terminology, historical discussion and proof):

Theorem 3. *Let T be a minimal interval exchange transformation. Let $e_T(n)$ denote the minimum length of an n-block of T^n. Let $c > 0$. Assume $n_j \in \mathbb{N}$ have the following two properties:*

(1) $\frac{n_{j+1}}{n_j} > 2$
(2) $e_T(n) > \frac{c}{n_j}$.

Let J be an n_i-block of T. There exist constants $C_1, C_2 > 0$ depending only on c such that for any points x, x' we have $\frac{1}{n_{i+L}}|\sum\limits_{j=1}^{n_{i+L}} \chi_J(S^j x) - \chi_J(S^j x')| < C_1 e^{-C_2 L}$.

Quantitative equidistribution results for interval exchanges have also been proven in [31], [12] and [2].

1.1. **Related results in other settings.**

Definition 1. *Given a dynamical system (X,T,μ), a sequence of sets $\{C_i\}$ is a* strong Borel Cantelli *sequence for T if*

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} \chi_{C_i}(T^i x)}{\sum_{i=1}^{N} \mu(C_i)} = 1$$

for almost every x.

This paper establishes that for almost every α any sequence of balls $B(\frac{1}{2}, r_i)$ so that $\{r_i\}$ is a Khinchin sequence is strong Borel Cantelli for R_α. If the rotation is badly approximable we may relax the condition to allow r_i just non-increasing and with divergent sum.

This question has been considered in systems of high complexity. Philipp [28] proved that for the Gauss map, or a β shift with the smooth invariant measure any seqeunce of intervals so that the sum of the measures diverge is strong Borel Cantelli. Dolgopyat [10] proved an analogous result for Anosov diffoemorphisms. Chernov-Kleinbock [9] proved a similar result for topological markov chains with a Gibbs measure: cylinders satisfying a certain nesting condition and so that the sum of their measures diverge are strong Borel Cantelli. To highlight the difference between our low complexity setting and the high complexity situation we remark that for every rotation α there is a sequence of the sets $[0,1] \cup [\frac{1}{4}, \frac{3}{4}]$ which is not strong Borel Cantelli.

1.2. **Outline of paper.** We prove our results following the outline of the strong law of large numbers.

In Section 2, we prove Theorem 4, the generalization of Theorem 1. The first key step is Proposition 2, which we prove in Section 2.2. This Proposition says that, in the presence of the diophantine assumption, a large part of the sum in the numerator of equation (1) can be broken up into sums over disjoint ranges for i in such a way that the resulting quantities are approximately independent.

Section 2.3 shows via this approximate independence result, that Theorem 1 is true if we ignore those terms in the sum which are not part of these roughly independent quantities. Then Section 2.4 treats the terms ignored in Section 2.3, showing that their contribution is negligible and finishing the proof.

We then prove Theorem 2 in two parts. In Section 3.1 we treat radius sequences $\{r_i\}$ where $\sup ir_i < \infty$. In Section 3.2 we treat the general case.

Section 4 proves the quantitative Boshernitzan criterion Theorem 3, which is used in the earlier sections.

There is an appendix that provides a treatment of the symbolic coding of an IET. This is well known material included for completeness, and to provide a reference for notation and terminology used elsewhere in the paper.

1.3. **Acknowledgments.** J. Chaika would like to thank B. Fayad and D. Kleinbock for encouraging me to pursue this question. We would like to thank J. Athreya, M. Boshernitzan, A. Eskin, H. Masur, R. Vance and W. Veech for helpful conversations. TJ. Chaika was partially supported by NSF grant DMS-1004372. We thank the anonymous referee for helpful suggestions that greatly improved the readability of the paper.

2. Proof of Theorem 1

2.1. **Setup and an outline of the proof.** In this section we introduce notation and terminology necessary to state and prove our main theorem for interval exchange transformations, Theorem 4. (Recall that this is a generalization of Theorem 1.) The main task is to introduce an analogue of the continued fraction expansion used to state Theorem 1. We also give a short outline of the proof of Theorem 4 as it will proceed in the following sections, and record a few Lemmas for future use.

Definition 2. *Given a vector $L = (l_1, l_2, ..., l_d)$ where $l_i \geq 0$, we obtain d subintervals of the interval $[0, \sum_{i=1}^{d} l_i)$:*

$$I_1 = [0, l_1), I_2 = [l_1, l_1 + l_2), ..., I_d = [l_1 + ... + l_{d-1}, l_1 + ... + l_{d-1} + l_d).$$

*Given a permutation π on the set $\{1, 2, ..., d\}$, we obtain a d-*Interval Exchange Transformation *(IET) $T\colon [0, \sum_{i=1}^{d} l_i) \to [0, \sum_{i=1}^{d} l_i)$ which exchanges the intervals I_i according to π. That is, if $x \in I_j$ then*

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

The points $\{\sum_{i=1}^{r} l_i\}$ are the discontinuities of T.

Recall the symbolic coding of an IET (Appendix A). Given an IET T, let $e_T : \mathbb{N} \to \mathbb{R}$ be defined as follows: $e_T(n)$ is the minimum distance between 2 discontinuities of T^n. If two discontinuities orbit into each other then $e_T(n)$ is defined to be 0. Since $T^{-1}(\{0,1\})$ is contained in the set of discontinuities we have that $e_T(n)$ is at most the measure of the smallest $(n-1)$-block (see Appendix A). Notice that e_T is a non-increasing function.

Fix $\xi > 0$. Let $n_i(\xi)$ be defined inductively by $n_{i+1} = \min\{2^k > n_i : e_T(2n_{i+1}) > \xi\}$. Let $a_i(\xi) = \frac{n_i}{n_{i-1}}$. Below, we will suppress ξ in our notation.

Theorem 4. *Let T be an IET so that for every $\epsilon > 0$ there exists $\xi > 0$, C so that $a_i \leq i^{\frac{4}{3}}$ for all but finitely many i and*

$$\limsup_{N\to\infty} \frac{1}{N}\Big(\sum_{i=1}^{N} \log a_i - \sum_{a_i<C}^{N} \log a_i\Big) < \epsilon. \tag{2}$$

Then for any Khinchin sequence $\{r_i\}$ we have

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} \chi_{B(\frac{1}{2}, r_i)}(T^i x)}{\sum_{i=1}^{N} 2r_i} = 1 \tag{3}$$

for almost every x.

The proof of Theorem 4 proceeds as follows. First, we split up the sum in the numerator of equation (3) into sums over disjoint sets of indices. Specifically, let

$$g_i(x) = \sum_{j=n_i}^{2n_i} \chi_{T^{-j}B(\frac{1}{2},r_j)}(x).$$

These sums account for much, but not all, of the sum in equation (3). We then show in Sections 2.2 and 2.3 that Theorem 4 holds if we ignore those terms in the sum not included in the g_i:

$$\lim_{N\to\infty} \frac{\sum\limits_{i=1}^{N} g_i(x)}{\sum\limits_{i=1}^{N} \int g_i} = 1. \tag{4}$$

We do this by showing that the g_i satisfy abstract properties that are sufficient. The abstract properties are given by the following version of the strong law of large numbers whose standard proof (given in Section 2.3) is included for completeness.

Proposition 1. *Let $H_i : [0,1] \to \mathbb{R}^+$ so that for all i there exists C_1, C_2:*

(1) $||H_i||_\infty < C_1$
(2) $\sum_{i=1}^\infty \int H_i = +\infty$
(3) $\sum_{j=i+1}^\infty \int H_j(x)H_i(x) - \int H_i(x) \int H_j(x) < C_2||H_i(x)||$.

Then

$$\lim_{N\to\infty} \frac{\sum_{i=1}^N H_i(x)}{\sum_{i=1}^N \int H_i(x)} = 1$$

for a.e. x.

Property (3) should be thought of as approximate independence of the H_i. Verifying it for g_i is the main work; this is shown in Section 2.2. This approximate independence for g_i comes via Lemma 3 from effective equidistribution (Theorem 3) and approximate T invariance (Lemma 4).

Having established equation (4), we complete the proof in Section 2.4 by showing that the times we have ignored are negligible. That is, for almost every x

$$\sum_{\{i\le N: \notin \cup_1^\infty [n_j,2n_j]\}} \chi_{B(\frac{1}{2},r_i)}(T^i x) = o(\sum_{i=1}^{N} \chi_{B(\frac{1}{2},r_i)}(T^i x)).$$

Observe that if $\{r_i\}$ is a Khinchin sequence then $\int g_i = O(\int g_j)$ for any $j < i$.

We conclude this section with the proof of a result used to control $\|g_i\|_\infty$ which we will frequently quote.

Lemma 1. $g_i(x) \le 1 + \frac{2n_i}{\xi} 2r_{n_i}$ *for all i.*

The proof relies on:

Lemma 2. *(Boshernitzan [4, Lemma 4.4]) If the orbits of the discontinuities of T are infinite and distinct then for any interval J of size $e_T(n)$ there exist integers $p \leq 0 \leq q$ (which depend on J) such that*

1. *$q - p \geq n$*
2. *T^i acts continuously on J for $p \leq i < q$*
3. *$T^i(J) \cap T^j(J) = \emptyset$ for $p \leq i < j < q$.*

Proof of Lemma 1. By Lemma 2's conclusion 3, if $T^j x, T^{j+r}x \in J$ then $|J| \leq e_T(r)$. Partition $B(\frac{1}{2}, r_{n_i})$ into intervals of size $e_T(n_i) \geq \frac{\xi}{2n_i}$. □

2.2. **Estimate on $\int g_j(x)g_i(x)$.** The goal of this section is to establish property (3) of Proposition 1 for the g_i. Let $i > j$.

Proposition 2. *There exists C so that for all j*

$$\sum_{i=j+1}^{\infty} \int g_i g_j - ||g_j||_1 ||g_i||_1 < C||g_j||_1.$$

C depends only on ξ.

We establish this proposition by showing that as i gets larger than j, most of the space can be partitioned into long orbits $\{T^k x\}_{k=0}^{m_{i,j}}$ so that for $0 \leq \ell, k \leq m_{i,j}$ we have $g_i(T^\ell x) \approx g_i(T^k x)$ and $\sum_{k=0}^{m_{i,j}} g_j(T^k x) \approx m_{i,j} \int g_j$. The next lemma shows how this implies approximate independence of g_i and g_j.

Lemma 3. *Assume h is a function satisfying $||h - h \circ T^i||_1 < \delta$ for $i \leq n$ and*

$$\big| n|J| - \{0 < i < n : T^i(x) \in J\} \big| < n\delta'.$$

Then

$$\int h\chi_J - |J| \int h \leq (|J| + \delta') \int h + \delta.$$

Proof. Let $e_i(x) = h(x) - h \circ T^i(x)$.

(5)
$$\int h(x)\chi_J(x) = \int \frac{1}{n} \sum_{i=1}^{n} (h \circ T^i(x) + e_i(x))\chi_J(x) \leq \int \frac{1}{n} \sum_{i=1}^{n} h \circ T^i(x)\chi_J(x) + \delta =$$
$$\int h(x) \frac{1}{n} |\{1 \leq i \leq n : T^{-i}(x) \in J\}| + \delta \leq (|J| + \delta')(\int h(x)) + \delta.$$

□

Notice in the first inequality we only use that $||\chi_J||_\infty \leq 1$ so by the same proof we obtain:

Corollary 1. Let $J_1, ..., J_k$ be intervals such that

$$\big| n|J_i| - \{0 < i < n : T^i(x) \in J_i\} \big| < n\delta'$$

for all $i \leq k$ and let h be a function so that $||h - h \circ T^i||_1 < \delta$ for all $0 \leq i \leq n$. Then

$$|\int h(x)\sum_{i=1}^k \chi_{J_i}(x) - \int h \int \sum_{i=1}^k \chi_{J_i}| < (\int \sum_{i=1}^k \chi_{J_i} + \delta')\int h + \delta ||\sum_{i=1}^k \chi_{J_i}||_\infty.$$

The first step in applying Corollary 1 to the g_i is establishing the approximate T invariance.

Lemma 4. *There exists C so that for every j*

$$\sum_{k=j+1}^{\infty} \max\{||g_k - g_k \circ T^s||_1 : 0 \leq s < n_{\frac{k+j}{2}}\} < C||g_j||_1.$$

Proof.

$$\sum_{i=M}^{N} \chi_{T^{-i}B(\frac{1}{2},r_i)}(x) - \sum_{i=M}^{N} \chi_{T^{-i}B(\frac{1}{2},r_i)}(T^s x) = \tag{6}$$

$$\sum_{i=M}^{M+s-1} \chi_{T^{-i}B(\frac{1}{2},r_i)}(x) - \sum_{i=N-s+1}^{N} \chi_{T^{-i-s}B(\frac{1}{2},r_i)}(x) + \sum_{i=M+s}^{N} \chi_{T^{-i}B(\frac{1}{2},r_i)}(x) - \chi_{T^{-i}B(\frac{1}{2},r_{i-s})}(x)$$

Since we assume that r_i is non-increasing the L_1 norm of this is at most

$$2sr_M + 2\sum_{i=M+s}^{N} r_{i-s} - r_i \leq 2(s+1)r_M.$$

Because our sequences are Khinchin sequences we obtain $\sum Cn_{\frac{k+i}{2}} r_{n_k} \leq n_i r_{2n_i} 2(1-\frac{1}{\sqrt{2}})^{-1}$. The lemma follows since $||g_i||_1 > n_i r_{2n_i}$. ☐

Theorem 3 establishes convergence of orbit sums for functions that are constant on intervals of continuity of T^M for appropriately chosen M, but we will need that some such a function is close to g_i. The next two lemmas show this.

Given a finite set $S \subset [0,1]$ let P_S be the finite partition of $[0,1]$ defined by connected components of $[0,1] \setminus S$.

Lemma 5. *If S is ϵ dense then there exists a function h which is constant on each element of P_S and whose L_1 difference from g_i is at most $2n_i\epsilon$. Moreover, h can be chosen so that $\|h\|_\infty \leq \|g_i\|_\infty$.*

Proof. It is straightforward to check that the characteristic function for any interval, χ_J is 2ϵ away from a function, ϕ, constant on elements of P_S and so that $\phi(x) \leq \chi_J(x)$ for all x. The lemma follows because g_i is the sum of n_i characteristic functions of intervals. ☐

Let S_k be the set of discontinuities of T^k. Recall that d is the number of intervals of our IET and ξ is the constant so that $e_T(n_i) > \frac{\xi}{n_i}$.

Lemma 6. *$S_{n_i+d(2-\log_2(\xi))}$ is at least $\frac{1}{n_i}$ dense.*

The lemma follows immediately from the following result, which is adapted to our situation. This result uses the *first return map*. Recall that if $S : X \to X$ is a dynamical system and $A \subset X$ then the first return map of S to A is $S|_A : A \to A$ by $S|_A(x) = S^{\min\{\ell>0: S^\ell x \in A\}}(x)$. The numbers $\min\{\ell > 0 : S^\ell x \in A\}$ are called return times. Recall that the first return map of a d-IET to an interval bounded by adjacent discontinuities of T^n is at most a d-IET and the return time is constant on each interval.

Sublemma 1. Let J be an m-block. Recall that d is the number of subintervals in our IET. Then there are at most $d(2 - \log_2(\epsilon))$ n_i between $\frac{1}{|J|}$ and m.

Proof. We show that there exist d numbers $k_1, ..., k_d$ between $\frac{1}{|J|}$ and m so that if $k_j < i < k_{j+1}$ then $e_T(i) < \frac{1}{k_{j+1}}$. The sublemma follows from this fact because by this assumption there can be at most $2 - \log_2(\xi)$ different n_ℓ between k_i and k_{j+1}. Consider J, an m-block. $J = [T^{-k}\delta, T^{-L}\delta')$ where δ, δ' are either $0, 1$ or discontinuities. Moreover $T^{-r}\delta'' \in J$ implies $r \geq m$. Observe that to each interval of $T|_J$ there is a corresponding interval of $T^{-1}|_J$ with the same length and return times: $r_1 \leq r_2 \leq ... \leq r_d$ (these are the k_i mentioned in the first sentence). There exists a return time, r_1 of size at most $\frac{1}{|J|}$. If $r_1 < m$ then the boundary point of this interval has to be in the orbit of δ, δ'. So it is either $T^{-s-k}\delta$ or $T^{-s-L}(\delta')$ where $s \leq r_i$. Pushing J forward by $k, L < n$ respectively we obtain two s-blocks, one of which returns to J at r_1 and one that is still outside. The part that is still outside will have no points return before r_2 and so its length is at most $\frac{1}{r_2}$. Inductively we have d disjoint d blocks contained in J, one of which does not have any points that return to J before r_{k+1} and so has length smaller than $\frac{1}{r_{k+1}}$. $\square$

The next lemma, whose proof is obvious, controls how different $\int g_i g_j$ can be from $\int h_i h_j$ if h_i is close to g_i and h_j is close to g_j.

Lemma 7. *If* $||f_1 - h_1||_1 < \epsilon_1$ *and* $||f_2 - h_2||_1 < \epsilon_2$ *then*

$$\int f_1(x)f_2(x) \leq \int h_1(x)h_2(x) + ||f_1||_\infty \epsilon_2 + ||f_2||_\infty \epsilon_1 + \epsilon_1\epsilon_2.$$

We now prove Proposition 2.

Proof of Proposition 2.

$$\sum_{i=j+1}^{\infty} |\int g_i g_j - \int g_j \int g_i| = \tag{7}$$

$$\sum_{i>j \text{ such that } n_i < r_{2n_j}^{-1}} |\int g_i g_j - \int g_j \int g_i| + \sum_{i>j \text{ such that } n_i \geq r_{2n_j}^{-1}} |\int g_i g_j - \int g_j \int g_i|.$$

Step 1: We estimate the first term. Let $t = 2n_j r_{2n_j}$, so by the Khinchin condition $r_k \leq \frac{t}{k}$ for all $k \geq 2n_j$. Notice that if $N > t^{-1}2n_j$ then $\frac{1}{N} < r_{2n_j}$. So the first term has at most $j < i \leq j + \log_2(t^{-1}) + 1$ summands. Moreover, for each such i, g_i is at most $t\log(2) + \frac{1}{2^{n_i}}$. We now show that there exists $\hat{C}$ so that $\int |g_i g_j - \int g_i \int g_j|$ is at most $\hat{C}t\log(2) + \frac{1}{2^{n_i}}$. By Lemma 1 $||g_i||_\infty \leq 1 + \frac{4}{\xi}t$. Since $\int g_i g_j \leq ||g_j||_1 ||g_i||_\infty$ it follows that there exists $\hat{C}$ so that the first summation is at most $\hat{C}t(\log(t^{-1})+1)$. Note: Because $r_k \leq \frac{r_1}{k}$, $\hat{C}$ can be chosen uniformly over all j.

Step 2: The second summand. To do this we will use Corollary 1 to show that g_i has little correlation with $f_{i,j}$, a function that is close to g_j. We will then apply Lemma 7 to show that g_i and g_j have little correlation.

We first build $f_{i,j}$. Let $S_{i,j}$ be the set of discontinuities of $T^{n_{\frac{3j+i}{4}}}$. By Lemma 6 we have that $S_{i,j}$ is $\frac{1}{n_{\frac{3j+i}{4}-r}}$ dense. Because $\{r_q\}$ is a Khinchin sequence, this is at most $2^{-\frac{i-j}{4}+r}r_{2n_j}$ for all $i > 4r$. By Lemma 5 for each $i > k_j + 4r$ there exists $f_{i,j}$ such that

$$||f_{i,j} - g_j||_1 \leq n_j r_{2n_j} 2 \cdot 2^{-\frac{i-j}{4}+r} < 2 \cdot 2^{-\frac{i-j}{4}+r}||g_j||_1 \tag{8}$$

and $\|f_{i,j}\|_\infty \leq \|g_j\|_\infty$.

Now we show that $f_{i,j}$ is fairly independent from g_i. If J is an $n_{\frac{3j+i}{4}}$ block then by Theorem 3 we have

$$|(|\{0 \leq i < n_{\frac{i+j}{2}} : T^i x \in J\}| - n_{\frac{i+j}{2}}|J|)| < C_1 C_2^{\frac{i-j}{4}}$$

for any x. Let $c_{j,i} = \max\{||g_i - g_i \circ T^r|| : 0 \leq r \leq n_{\frac{i+j}{2}}\}$. Since $f_{i,j}$ is the sum of k disjoint characteristic functions of $n_{\frac{3j+i}{4}}$ blocks then by Corollary 1

$$|\int g_i(x) f_{i,j}(x) - \int g_i(x) \int f_{i,j}(x)| \leq C_1 C_2^{\frac{i-j}{4}} \int g_i(x) + c_{j,i}\|f_{i,j}\|_\infty.$$

By Lemma 4 we have that $\sum_{i=j+1}^{\infty} c_{j,i} < C||g_j||_1$. Lemma 1 and the Khinchin condition imply that there exists $\tilde{C}$ for that $\|g_k\| \leq \tilde{C}$ for all k. By Lemma 5 we have $\|f_{i,j}\|_\infty \leq \|g_j\|$ and it follows that $\|f_{i,j}\|_\infty < \tilde{C}$. Also by the Khinchin condition $||g_i||_1 \leq ||g_j||_1$. Combining these facts we obtain

$$\sum_{i=j}^{\infty} |\int g_i(x) f_{i,j}(x) - \int g_i(x) \int f_{i,j}(x)| \leq \hat{C}||g_j||_1. \tag{9}$$

So by Lemma 7 we have

$$\int g_j g_i - \int g_j \int g_i \leq \int f_{i,j} g_i - \int f_{i,j} \int g_i + 2||g_i||_\infty 2 \cdot 2^{-\frac{i-k_j}{4}+r}||g_j||_1$$

where $||g_i||_\infty < \tilde{C}$.

Combining Equations 9 and 8, the proposition follows. □

2.3. **Abstract setting: Proof of Propostion 1.** We prove Proposition 1 in two steps, Lemmas 10 and 11 below. Let H_i be as in Proposition 1. Let $m_0 = 0$ and m_k be defined inductively by $m_{k+1} = \min\{i : \sum_{m_k+1}^{m_i} \int |H_i| \geq 1\}$. Let $F_i = H_i - \int H_i$. Observe that F_i satisfies the following:

(1) $\int F_i = 0$
(2) $||F_i||_\infty < C_2$
(3) $\sum_{j=i+1}^{\infty} \int F_j(x) F_i(x) < C_1' ||F_i(x)||_1$

To prove Propostion 1 we use the following two classical results:

Lemma 8. *(Chebyshev's inequality) Let R be a random variable with $\int R d\mu = 0$ and finite variance then $\mu(\{\omega : R(\omega) > c\}) \leq \frac{\int R^2 d\mu}{c^2}$.*

Lemma 9. *(Borel-Cantelli) If $A_1, \ldots$ are m measurable sets and $\sum_{i=1}^{\infty} m(A_i) < \infty$ then $m(\{x : x \in A_i$ for infinitely many $i\}) = 0$.*

Using these, we prove:

Lemma 10.
$$\lim_{N\to\infty} \frac{\sum_{i=1}^{m_{N^2}} F_i(x)}{N^2} = 0$$
for a.e. x.

Proof. We first show that there exists $\tilde{C} > 0$ so that

$$\int (\sum_{i=1}^{m_M} F_i(x))^2 < \tilde{C} M.$$

The left hand side is $\int \sum_{i=1}^{m_M} F_i(x)^2 + 2\sum_{j=i+1}^{m_M} F_i(x)F_j(x)$. By assumption (3) we have $2|\sum_{j=i+1}^{m_M} f_i(x)f_j(x)| \leq \sum_{i=1}^{m_M}(2C_1') \int |F_i(x)|$. Also by assumption (2), $\sum_{i=1}^{m_M} \int |F_i(x)|^2 \leq \sum_{i=1}^{m_M} \|F_i\|_1 \max\{C_2^2, 2\}$.

Now by Chebyshev's inequality for each N we have

$$\lambda(\{x : |\sum_{i=1}^{m_{N^2}} F_i(x))| > \delta N^2\}) < \frac{\tilde{C}}{\delta^2 N^2}$$

for each δ. These sums converge and so for any $\delta > 0$ we have that for almost every x

$$\limsup_{N\to\infty} \frac{|\sum_{i=1}^{m_{N^2}} F_i(x)|}{N^2} \leq \delta.$$

□

Let $\delta > 0$ be given and let

$$A_N = \{x : \max_{r \leq m_{(N+1)^2}} |\sum_{i=m_{N^2}}^{r} H_i(x)| > \delta N^2\}.$$

Lemma 11. $\sum_{i=1}^{\infty} \lambda(A_n) < \infty$.

Proof. Observe that since

$$\sum_{i=m_{N^2}+1}^{m_{(N+1)^2}} ||H_i||_1 < 2N + 1 + C_2$$

and $F_i(x) = H_i(x) - ||H_i||_1$, if

$$\max_{r \leq m_{(N+1)^2}} |\sum_{i=m_{N^2}}^{r} H_i(x)| > \delta N^2 \text{ then } \sum_{i=m_{N^2}}^{m_{(N+1)^2}} F_i(x) > \delta N^2 - 2N + 1 + C_2.$$

Analogously to the first step of Lemma 10 we have that $\int (\sum_{m_n+1}^{m_{n+1}} |F_i(x)|)^2 \leq C''$. It follows that $\int \max_{r \leq m_{(N+1)^2}} |\sum_{i=m_{N^2}}^{r} H_i(x)| < C'''N$ and by Chebyshev's inequality $\lambda(A_N) < C'''N^{-3}$.

□

Proof of Proposition 1. By Lemma 10 it suffices to show

$$\limsup_{N\to\infty} \left|\frac{\max_{r<m_{(N+1)^2}} \sum_{i=m_{N^2}+1}^{r} F_i(x)}{N^2}\right| = 0$$

for almost every x. This follows by Lemma 11. □

2.4. **Controlling the omitted terms.** We restrict our attention to $\sum_{j\notin\cup[n_i,2n_i]} \chi_{B(\frac{1}{2},r_i)}(T^i x)$, the terms omitted in our consideration of g_i.

Let $\beta_i = \sum_{j=2n_i+1}^{n_{i+1}} \chi_{B(\frac{1}{2},r_j)}(T^j x)$. By the assumption on T in Theorem 4, if ξ is small enough then for most i, β_i is the zero function.

Proposition 3. *For any $\epsilon > 0$ and almost every x we have $\sum_{i=1}^{k} \beta_i(x) < \epsilon \sum_{i=1}^{n_k} 2r_i$ for all sufficiently large k.*

The first step in the proof is a version of Lemma 1 for our current setting. Let $s_{i+1} = \frac{n_{i+1}}{n_i}$.

Lemma 12. $\sum_{j=2n_i}^{n_{i+1}} \chi_{B(0,r_j)}(T^j(x)) \leq 6\xi^{-1}\sqrt{2n_i r_{2n_i}}\sqrt{s_{i+1}}$.

Proof. We prove the lemma by the following trivial estimate:

$$\max_x \sum_{j=2n_i}^{n_{i+1}} \chi_{B(0,r_j)}(T^j(x)) \leq \max_x \sum_{j=2n_i}^{u} \chi_{B(0,r_j)}(T^j(x)) + \max_x \sum_{j=u}^{n_{i+1}} \chi_{B(0,r_u)}(T^j(x)).$$

By Lemma 1 there are at most $2\xi^{-1}2n_i r_{2n_i}$ hits to an interval of size $2r_{2n_i}$ on an orbit of length $2n_i$. Let $t = 2n_i r_{2n_i}$, so $r_{2n_i} = \frac{t}{2n_i}$ and $r_j \leq \frac{t}{j}$ for all $j > 2n_i$ by the Khinchin condition. After $\sqrt{t}\sqrt{s_{i+1}}$ sets of $2r_{2n_i}$ by the Khinchin condition the interval has decayed to at most $\frac{\sqrt{t}}{\sqrt{s_{i+1}}n_i}$. Since the first n_{i+1} elements of the orbit are $\frac{\xi}{n_{i+1}} = \frac{\epsilon}{s_{i+1}n_i}$ separated there can only be $\frac{\sqrt{t}\sqrt{s_{i+1}}}{\xi}$ of them in this interval. Let $u = \max\{\sqrt{t}, 1\}\sqrt{s_{i+1}}2r_{2n_i}$ and the lemma follows. □

The next step is the following probabilistic result, which is an analogue of Proposition 1:

Lemma 13. *Let $H_i : [0,1) \to \mathbb{R}^+$ be a family of functions so that for every $\epsilon > 0$ there exists M so that*

1. $\sum_{i=1}^{N} ||H_i||_1 < \epsilon C_N$ *for all $N > M$.*
2. $\max_{i<N,x}\{H_i(x)\} < C_0 C_N^{\frac{2}{3}}$
3. $\sum_{N>j>i} \int H_i(x)H_j(x) < C_2 C_N^{\frac{2}{3}} ||H_i||_1$.

Then for almost every x we have $\frac{\limsup_{N\to\infty} |\sum_{i=1}^{N} H_i(x)|}{C_N} = 0$.

Let $R_i = H_i - \int H_i$.

Proof. As before we compute the variance. $\int(\sum_{i=1}^{N} R_i(x))^2 \leq \epsilon C_0 C_N^{\frac{5}{3}} + C_3 C_N^{\frac{4}{3}}$. This follows because $||h||_2^2 \leq ||h||_1 ||h||_\infty$ and because of Condition 3. By Chebyshev's inequality

$$\lambda(\{x : \sum R_i(x) > \epsilon C_N\}) \leq C_6 C_N^{-\frac{1}{3}}.$$

Let $k_r = \min\{M : C_N > r \text{ for all } N > M\}$. By the Borel-Cantelli Lemma it follows that

$$\frac{\limsup_{N\to\infty} |\sum_{i=1}^{k_{N^4}} R_i(x)|}{N^4} \leq \epsilon.$$

Now consider $\sum_{i=k_{N^4}}^{k_{(N+1)^4}} R_i(x)$. By the definition of R_i we have

$$\max_{L<k_{(N+1)^4}} \sum_{i=k_{N^4}}^{L} R_i(x) \leq \sum_{i=k_{N^4}}^{k_{(N+1)^4}} R_i(x) + ||H_i||_1.$$

So the square of the L_2 norm is at most $C_7 N^4 (N^4)^{\frac{2}{3}}$. By Chebyshev's inequality

$$\lambda(\{x : \max_{L<k_{(N+1)^4}} |\sum_{i=k_{N^2}}^{L} R_i(x)| > \epsilon N^4\}) \leq C_8 \frac{N^4 N^{\frac{8}{3}}}{N^8} = C_8 N^{-\frac{4}{3}}$$

where C_8 depends on ϵ. By the Borel-Cantelli Lemma almost every x can have $\sum_{j=k_{n^4}}^{r} R_i(x) < \epsilon n^4$ where $r < k_{(n+1)^4}$ only finitely many times. So considering N as $m^4 + i$ where m is the largest positive integer such that $m^4 \leq N$ we obtain

$$\frac{\limsup_{N\to\infty} |\sum_{i=1}^{N} R_i(x)|}{C_N} \leq 2\epsilon$$

for almost every x. □

Proof of Proposition 3. It suffices to show that β_i satisfy the assumption of Lemma 13 with $C_N = \sum_{i=1}^{n_N} 2r_i$. Conditions (1) and (2) follows from our diophantine assumption on T, the definition of β_i and Lemma 12. Condition (3) follows analogously to Proposition 2. □

We are now ready to complete the proof of Theorems 1 and 4.

Proof of Theorems 1 and 4. By Proposition 1 we have

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} g_i(x)}{\sum_{i=1}^{N} \int g_i} = 1.$$

Choose $\delta > 0$. There exists $\xi > 0$ so that with g_i defined for this ξ we have

$$\liminf_{N\to\infty} \frac{\sum_{i=1}^{N} \int g_i}{\sum_{i=1}^{2n_N} 2r_i} > 1 - \delta.$$

By Lemma 1 we have that

$$\liminf_{N\to\infty} \frac{\sum_{i=1}^{N} \chi_{B(0,b_i)}(R_\alpha^i x)}{\sum_{i=1}^{N} 2r_i} > 1-\delta.$$

Now by Lemmas 1 and 12

$$\limsup_{N\to\infty} \frac{\sum_{i=1}^{n_N} \chi_{B(\frac{1}{2},r_i)} R_\alpha^i x}{\sum_{i=1}^{n_N} 2r_i} = \limsup_{N\to\infty} \frac{\sum_{i=1}^{N} g_i + \sum_{i=1}^{n_N} \beta_i}{\sum_{i=1}^{n_N} 2r_i}.$$

By Proposition 3

$$\limsup_{N\to\infty} \frac{\sum_{i=1}^{n_N} \beta_i}{\sum_{i=1}^{n_N} 2r_i} < \epsilon$$

for all $\epsilon > 0$. By Propositions 1 and 2 we have

$$\limsup_{N\to\infty} \frac{\sum_{i=1}^{N} g_i}{\sum_{i=1}^{n_N} 2r_i} = 1$$

and so the theorem follows. □

3. Proof of Theorem 2

We now turn to the proof of Theorem 2. Recall that in this theorem the assumption that α is badly approximable allows us to consider a wider class of radius sequences $\{r_i\}$. As in Section 2, we will state and prove a generalization the theorem to the case of interval exchange transformations. Using the notation developed in section 2.1, this generalization is:

Theorem 5. *Let T be an IET so that there exists $\sigma > 0$ with $e_T(n) > \frac{\sigma}{n}$ for all n. Then for any decreasing sequence $\{r_i\}$ with divergent sum we have:*

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} \chi_{B(y,r_i)}(T^i x)}{\sum_{i=1}^{N} 2r_i} = 1$$

for almost every x.

Let σ be a constant so that $e_T(n) > \frac{\sigma}{n}$ for all n. Let $g_i = \sum_{j=2^i}^{2^{i+1}-1} \chi_{B(\frac{1}{2},r_i)}(T^i x)$.

The proof we provide is complicated by the fact that without the Khinchin condition on r_i it is possible for $g_j \gg g_i$ for some $j > i$. This difficulty is handled for most values of i by appealing directly to Theorem 3. We must then show that the remaining indices, which are not handled by our appeal to Theorem 3, make negligible contributions.

The outline of this section is as follows. We break up our indices into two disjoint sets according to a (fixed, large) parameter M. Section 3.1 deals with those times i such that $ir_i < M$. The proof in this section is similar to that in Section 1 but simpler because we do not need to worry about the issues of Section 2.4. Then in Section 3.2 we treat the times i such that $ir_i \geq M$. We partition them into some blocks where we may apply Theorem 3 and some remaining blocks, whose contributions we show are negligible. Lemma 16 accomplishes the partitioning the blocks of i where ir_i is large, Lemma 18 applies Theorem 3 and Corollary 2 controls the size of the blocks where we can not apply Theorem 3. We note that the arguments in Section 3.1 work for any value of M. It is for the proofs in Section 3.2 that we have to chose a sufficiently large value of M.

Throughout this section, in an abuse of notation, r_{C^L} denotes $r_{\lfloor C^L \rfloor}$.

3.1. $\boldsymbol{ir_i}$ **small.** For this subsection we treat $ir_i < M$.

Proposition 4. *Let C, M be given. Let $E = \{i : r_{C^{i+1}} < \frac{M}{C^{i+1}}\}$. Let $g_i = \sum_{j=C^i}^{C^{i+1}} \chi_{B(\frac{1}{2}, r_j)}(T^j x)$. If $|E| = \infty$ then*

$$\lim_{N \to \infty} \frac{\sum_{i \in E}^{N} g_i(x)}{\sum_{i \in E}^{N} \int g_i} = 1.$$

We first state the appropriate version of approximate T invariance, which follows by a straightforward modification of Lemma 4 and is left to the reader.

Lemma 14. *There exist D_1, D_2 that depend only on C, σ and M such that if $i < j$ then*

$$\max_{k < 2^{\frac{i+j}{2}}} ||g_j - g_j \circ T^k|| < D_1 D_2^{j-i}.$$

Proof of Proposition 4. Let $E = \{a_1 < a_2, ...\}$ be infinite. We show $H_i = g_{a_i}$ satisfy the assumptions of Proposition 1. By our assumption on r_{a_i} and Lemma 1 we have $||g_{a_i}||_\infty < 2M\sigma^{-1}$ and so condition 1 is satisfied. The assumption that $|E| = \infty$ implies condition 2. Condition 3 follows verbatim to the proof of Proposition 2 in Section 2.2 with Lemma 14 in place of Lemma 4. Indeed if $2^{-k} < r_{2i}$ then

$$\int g_i g_j \leq C_1 C_3^{j-k} \int g_i \int g_j \leq C_1' C_3^{j-k} M \int g_i.$$

□

3.2. $\boldsymbol{ir_i}$ **big.** When $ir_i \geq M$ we want to use the next lemma, which requires M sufficiently large:

Lemma 15. *Let T be of constant type and $C > 1$.*

$$\lim_{M \to \infty} \limsup_{j \to \infty} \left| \frac{1}{C^{j+1} - C^j} \sum_{i=C^j}^{C^{j+1}} \chi_{B(\frac{1}{2}, \frac{M}{C^{j+1}})}(T^i x) - 1 \right| = 0.$$

Proof. Because T is of constant type, fixing C, k, by Lemma 6 for large enough M, $B(\frac{1}{2}, \frac{M}{C^{j+1}})$ can be approximated up to an ϵ proportion by C^{j-k}-blocks (of T). We can apply Theorem 3 with $L = k$. By choosing k large enough (given C, σ) we can treat the C^{j-k}-block. The rest of $B(\frac{1}{2}, \frac{M}{C^{j+1}})$ is hit at most $(2\sigma^{-1}+1)\epsilon \frac{M}{C^{j+1}}(C^{j+1} - C^j)$ times by Lemma 1. The lemma follows. □

The next lemma lets us split up the natural numbers into chunks where we appeal to Proposition 4, chunks where we can apply Lemma 15 (see Lemma 18) and a small remaining piece that we show is negligible (see Corollary 2). Throughout the remainder of this section $C > 1$ should be thought of as very close to 1.

Lemma 16. *Define*

$$G_{C,\rho,M}=\{j\in\mathbb{N}: r_{C^{j+1}}\geq \frac{M}{C^{j+1}} \text{ and } r_{C^j}\leq \rho r_{C^{j+1}}\}$$

and

$$B_{C,\rho,M}=\{j\in\mathbb{N}\setminus G_{C,\rho,M}: r_{C^{j+1}}\geq \frac{M}{C^{j+1}}\}.$$

For any $\epsilon>0$, $\rho<1$ there exists $C>1$ so that for any $\{r_i\}\subset\mathbb{R}^+$ where $r_i\geq r_{i+1}$ we have

$$\limsup_{N\to\infty}\frac{\sum_{i:i\in[C^j,C^{j+1}]\ s.t.j\in B_{C,\rho,M}}^N r_i}{\sum_{i=1}^N r_i}<\epsilon$$

for all $M>2\max\{1,r_1\}$.

Note if S is a set then $\sum_{i\in S}^N r_i$ denotes $\sum_{i\in S\cap[0,N]}$.

Proof. Let $\rho<1$ be given. By assumption $r_1<\frac{M}{2}$. Enumerate $B_{C,\rho,M}=b_1,\dots$. Let S_1 be the $\lceil\frac{1}{8}\log_C(\frac{1}{\rho})\rceil$ largest indices $j<b_1$ and so that $r_{C^j}\in[\sqrt{\rho}C^{b_1}r_{C^{b_1}},C^{b_1}r_{C^{b_1}}]$. Since $r_{C^{b_1}}>\frac{M}{C^{b_1}}$ by our assumption on M such a set exists. Inductively, given $S_1,\dots,S_k$ let S_{k+1} be the $\lceil\frac{1}{3}\log_C(\frac{1}{\rho})\rceil$ largest indices in

$$\{1,\dots,b_{k+1}\}\setminus\big(B_{C,\rho,M}\cup_{i=1}^k S_i\big).$$

Observe that if $C>1$ is close enough to 1 then such a set exists. Indeed, $b_n\geq(n\frac{1}{2}\log_C\frac{1}{\rho})+\log_C(\frac{M}{2})-1$.

Given ρ, if $C>1$ is close enough to 1 we have $\epsilon\sum_{i\in S_\ell}\sum_{j=C^i}^{C^{i+1}}2r_j>\sum_{j=C^{b_\ell}}^{C^{b_\ell+1}}2r_j$. □

To control $\sum_{k=C^i}^{C^{i+1}}\chi_{B(\frac{1}{2},r_k}(T^kx)$ where $i\in B_{C,\rho,M}$ we need the following result.

Lemma 17. *Let T be an IET of constant type, $\{r_i\}$ nonincreasing. Then there exists C' depending only on the constant type so that*

$$\sum_{j=C^i}^{C^{i+1}}\chi_{B(\frac{1}{2},r_j)}(T^jx)<C'\Big(\sum_{j=C^i}^{C^{i+1}}2r_j+\log(\frac{r_{C^i}}{r_{C^{i+1}}})+1\Big).$$

Proof. By Lemma 2 if $m>n$ then $T^nx\in B(a,r_n)$ and $T^mx\in B(a,r_m)$ then $e_T(m-n)\geq 2r_n$. Let $t_1<\dots<t_k\in[2^i,2^{i+1}]$ be numbers so that $T^{t_j}(x)\in B(a,r_{t_j})$. If $2r_{t_{i+1}}>r_{t_i}$ then $\sum_{j=t_i}^{t_{i+1}}2r_j\geq\frac{\sigma}{4}$. So $1+\sum_{i=t_1}^{t_k}2r_i\leq(k-\log_2(\frac{r_{C^i}}{r_{C^{i+1}}}))\frac{\sigma}{4}$. □

Corollary 2. For every $\epsilon>0$ and $\rho<1$ there exists C so that for all x, and $M>2\max\{1,r_1\}$ we have

$$\frac{\sum_{i\in B_{C,\rho,2M}}^N\chi_{B(\frac{1}{2},r_i)}T^ix}{\sum_{j=1}^N\sum_{i=C^j}^{C^{j+1}}2r_i}<\epsilon.$$

Lemma 18. *For any* $\epsilon > 0$, $C > 1$ *there exists* $M_0 > 1$ *so that if* $M > M_0$, $\rho = 1 - \frac{1}{16}\sigma\epsilon$ *then for any* $j \in G_{C,\rho,M}$

$$\frac{\sum_{i=C^j}^{C^{j+1}} \chi_{B(\frac{1}{2},r_i)}(T^i x)}{\sum_{i=C^j}^{C^{j+1}} 2r_i} \in [1-\epsilon, 1+\epsilon].$$

The lemma follows by using Lemmas 1 and 15 to show that

$$\frac{\max |\{C^j \leq i < C^{j+1} : T^i x \in B(\frac{1}{2}, r_i)\}|}{\min |\{C^j \leq i \leq C^{j+1} : T^i x \in B(\frac{1}{2}, r_{C^{j+1}})\}|} < 1 + \epsilon.$$

Proof. We assume $\epsilon < 1$. Following Lemma 15 choose L so that

$$\limsup_j \Big|\frac{1}{C^{j+1} - C^j} \sum_{i=C^j}^{C^{j+1}} \chi_{B(\frac{1}{2}, \frac{L}{C^{j+1}})}(T^i x) - 1\Big| < \frac{\epsilon}{8}.$$

Let $M_1 = 2L$. By Lemma 15, if $j \in G_{C,\rho,M}$, we have

$$\min \sum_{i=C^j}^{C^{j+1}} \chi_{B(\frac{1}{2}, r_{C^{j+1}})}(T^i x) \geq (1 - \frac{\epsilon}{8}) 2r_{C^j}\rho(C-1).$$

Let us consider $B(\frac{1}{2}, r_i) \setminus B(\frac{1}{2}, r_{C^{j+1}})$ when $i \geq C^j$. This is contained in two fixed intervals of size at most $(1-\rho)r_{C^j}$. By Lemma 1 we have

$$\max_x |\{C^j \leq i < C^{j+1} : T^i x \in B(\frac{1}{2}, r_i) \setminus B(\frac{1}{2}, r_{C^{j+1}})\}| \leq \\ 2\sigma^{-1} 2(1-\rho) r_{C^j}(C^{j+1} - C^j) \leq \frac{\epsilon}{8} 2r_{C^j}(C^{j+1} - C^j).$$

So $\frac{\max |\{C^j \leq i < C^{j+1} : T^i x \in B(\frac{1}{2}, r_i) \setminus B(\frac{1}{2}, r_{C^{j+1}})\}|}{\min |\{C^j \leq i \leq C^{j+1} : T^i x \in B(\frac{1}{2}, r_{C^{j+1}})\}|} < \frac{1+2\frac{\epsilon}{8}}{1-\frac{\epsilon}{8}} \leq \frac{\epsilon}{2}$ by our assumption that $\epsilon < 1$. This establishes the lemma. □

We are now ready to prove Theorems 2 and 5.

Proof of Theorems 2 and 5. It suffices to show that for all $\delta > 0$ there exists $C > 1$ so that

$$\liminf_{N \to \infty} \frac{\sum_{j=1}^{N} \sum_{i=C^j}^{C^{j+1}} \chi_{B(\frac{1}{2}, r_i)} T^i x}{\sum_{j=1}^{N} \sum_{i=C^j}^{C^{j+1}} 2r_i} > 1 - \delta$$

and

$$\limsup_{N \to \infty} \frac{\sum_{j=1}^{N} \sum_{i=C^j}^{C^{j+1}} \chi_{B(\frac{1}{2}, r_i)} T^i x}{\sum_{j=1}^{N} \sum_{i=C^j}^{C^{j+1}} 2r_i} < 1 + \delta.$$

Choose $\rho = 1 - \frac{1}{8}\sigma\epsilon$. Following Corollary 2, choose C for this ρ and $\epsilon = \frac{\delta}{2}$. Following Lemma 18, choose M for these ρ, C, ϵ. So we have

$$\limsup_{N\to\infty} \frac{\sum_{i\in G_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} \chi_{B(\frac{1}{2},r_k)}(T^k x)}{\sum_{i\in G_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} 2r_k} < 1+\delta$$

and

$$\liminf_{N\to\infty} \frac{\sum_{i\in G_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} \chi_{B(\frac{1}{2},r_k)}(T^k x)}{\sum_{i\in G_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} 2r_k} > 1-\delta.$$

Proposition 4 implies

$$\lim_{N\to\infty} \frac{\sum_{i\notin G_{C,\rho,M}\cup B_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} \chi_{B(\frac{1}{2},r_k)}(T^k x)}{\sum_{i\notin G_{C,\rho,M}\cup B_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} 2r_k} = 1$$

for almost every x. By Corollary 2

$$\limsup_{N\to\infty} \frac{\sum_{i\in B_{C,\rho,M}}^N \sum_{k=C^i}^{C^{i+1}} \chi_{B(\frac{1}{2},r_k)}(T^k x)}{\sum_{k=1}^{C^N} 2r_k} < \frac{\delta}{2}$$

which completes the proof. □

4. Quantitative Boshernitzan's criterion

This section uses Appendix A.

Theorem 6. *(Boshernitzan [5]) Let $S : X \to X$ be the left shift acting minimally on a symbolic dynamical system. Let μ be an S-invariant measure. Let ϵ_n be the μ measure of the smallest cylinder set of length n. If there exists a constant c such that for infinitely many n, $\epsilon_n \geq \frac{c}{n}$, then the left shift is μ uniquely ergodic.*

This was proved for IETs by Veech [30], in which case the invariant/ergodic measure is Lebesgue. Masur [23] established the analogous, in fact stronger, result for flows on flat surfaces.

Let n_i be an increasing sequence of integers such that $\epsilon_{n_i} > \frac{c}{n_i}$ and $n_i > 10n_{i-1}$. Let us recall Theorem 3:

Theorem. *Let S, μ, ϵ_n be as in Theorem 6. Let b be a block of length n_i. There exist constants C_1, C_2 depending only on c such that for any words w, w' we have* $\frac{1}{n_{i+L}} |\sum_{j=1}^{n_{i+L}} \chi_b(S^j w) - \chi_b(S^j w')| < C_1 e^{-C_2 L}$.

This is a quantitative version of Boshernitzan's criterion because it tells how quickly any orbit equidistributes. Quantitative ergodicity statements for IETs and flows have been profitably studied with deep results in [12], [31] and [2].

For ease of notation we treat the case where $n_i = 1$; the general case is the same.

Let $B \subset \{1, ..., d\}$.

Let $a_n(w|B) = \frac{|\{i\leq n: w_i \in B\}|}{n}$.

Let $M_n[B] = \max_w a_n(w|B)$ and $m_n[B] = \min_w a_n(w|B)$. The next lemma is similar to a key step in [5].

Lemma 19. *If $\epsilon_n > \frac{c}{n}$ then*

$$\mu\Big(\big\{w : a_n(w|B) \in \big[\frac{3}{4}m_n[B] + \frac{1}{4}M_n[B], \frac{1}{4}m_n[B] + \frac{3}{4}M_n[B]\big]\big\}\Big)$$
$$\geq c\big(\frac{1}{4}m_n[B] + \frac{3}{4}M_n[B] - (\frac{3}{4}m_n[B] + \frac{1}{4}M_n[B])\big).$$

Proof. Let $u_1, ..., u_n$ be an allowed n block with exactly $nm_n[B]$ occurrences of a letter in B and $v_1, ..., v_n$ be an allowed n block with exactly $nM_n[B]$ occurrences of a letter in B. By minimality there is $w = ..., u_1, ..., u_n, ..., v_1, ..., v_n, ...$ Consider the successive blocks of length n formed by moving one place along ω. At each step the change in $a_n(\cdot|B)$ can be at most $\frac{1}{n}$. So there needs to be at least

$$n(\frac{1}{4}m_n[B] + \frac{3}{4}M_n[B] - (\frac{3}{4}m_n[B] + \frac{1}{4}M_n[B]))$$

different n blocks with $a_n(\cdot|B)$ in our desired range (these blocks are different by the fact that $a_n(\cdot|B)$ assigns them different values). The lemma follows by our assumption on ϵ_n. □

The next proposition is similar to results used in [30].

Proposition 5. *If $\epsilon_{2n} > \frac{c}{2n}$ then $[0, 1)$ is the union of at most $3d$-Rokhlin towers of height between n and $2n$, and with every level of μ-measure at least $\frac{c}{2n}$.*

See the last paragraph of Appendix A for the definition of Rokhlin tower.

Proof. Build disjoint towers with n levels such that that their bases are intervals bounded by discontinuities of T^n. Get a maximal collection of such towers. Every point is within n forward iterates of one of these towers. Whenever one can disjointly continue a pre-existing tower by forward iterates, do so. These towers will have height at most $2n$. If this is not possible (that is extending the tower hits a discontinuity of T before it is exhausted) then split the levels of the tower so that it can continue. The new subintervals will be bounded by discontinuities of T^{2n} (because they hit the discontinuity in at most $n + n$ steps). □

Given n_i let $\mathcal{R}_i$ be a collection of towers as in Proposition 5.

Remark 1. Notice that by construction each level has μ-measure $O(\frac{1}{n_i})$.

Lemma 20. *Let C_i be the set of towers in $\mathcal{R}_i$ which have at least $\frac{c^{2i+1}}{16^i}$ occurrences of the symbol* 1. *Then $\mu(C_{i+1}) \geq \min\{1, (1 - \frac{c^2}{16})\mu(C_i) + \frac{c^2}{8}\}$.*

Proof. Choose c so that the measure of the cylinder set defined by 1 is at least c. Let C_0 be this cylinder set. We assume $\mu(C_{i+1}) < 1$ and show that $\mu(C_i^c \cap C_{i+1}) \geq \frac{c^2}{8}$. Consider the words of length n_{i+1} as being concatenations of towers from $\mathcal{R}_i$ (i.e. words of length n_i). We now begin an argument similar to Lemma 19. Consider a word of length n_{i+1}, with the minimal occurrences of blocks in C_i, v and a word with the maximal occurrences of blocks in C_i, u. There is a word between them $vwu = V$. In this word consider the latest index j so that $v_j, ..., v_{j+n_{i+1}-1}$ has at most $\frac{1}{4}\mu(C_i)$ of its blocks in C_i. Every subsequent occurrence of a block from C_i^c gives points that are in S_i^c but not in C_i. This is because the word of length n_{i+1} that they start have at least $\frac{1}{4}\mu(S_i)$ of their blocks in C_i. Moreover each block has between n_i and $2n_i$ letters. By an estimate from Lemma 19 we have the measure bound.

Now we show that the measure of the set of points in S_i and not in C_{i+1} is at most $\mu(C_i) - \frac{c^2}{16}$. Let $h_i : S_i \to \mathbb{N}$ by $h_i(x) = \min\{n > 0 : S^n x \in S_i\}$. By Kac Lemma (see for example [21, Theorem 3.6]) $\int_{C_i} h_i d\mu = \mu(X) = 1$. if $x \in C_i \setminus C_{i+1}$ then $\sum_{j=0}^{\frac{c^3 n_{i+1}}{16}} h_i(S|_{S_i}(x)) \geq n_{i+1}$. Thus $\mu(C_i \setminus C_{i+1}) \leq \frac{c^2}{1}6$. □

As a straightforward consequence we obtain:

Corollary 3. There exist r and $\delta > 0$ depending only on c such that any block of length n_{i+r} contains at least $\delta\epsilon_{n_i} n_{i+r}$ disjoint occurrences of a block of length n_i.

Proof of Theorem 3. We prove this by induction assuming it is true for $L = kr$ and proving it for $L = (k+1)r$. To each Rokhlin tower given by Proposition 5 for n_{i+kr} give a symbol. Given an $n_{i+(k+1)r}$ block write it as a concatenation of these symbols (plus a prefix and suffix of length at most n_{i+kr}). Consider the symbols that correspond to the n_{i+kr} towers that have the maximal and minimal frequency of a given letter. Denote these frequencies by Ξ and ξ, respectively. By corollary 3 each $n_{i+(k+1)r}$ block has δ proportion of its letters coming from each of these towers. So the frequency of each symbol is between $\delta\Xi + (1-\delta)\xi$ and $(1-\delta)\Xi + \delta\xi$. The theorem follows by induction. □

Appendix A. Symbolic coding

We use the symbolic coding of interval exchange transformations heavily. This section also shows the well known and useful fact that IETs are basically the same as (measure conjugate to) continuous maps on compact metric spaces. For concreteness assume that $\sum_{i=1}^{d} l_i = 1$.

Let $\tau\colon [0,1) \to \{1,2,...,d\}^{\mathbb{Z}}$ by $\tau(x) = ..., a_{-1}, a_0, a_1, ...$ where $T^i(x) \in I_{a_i}$.

Fixing a point x, that is not in the orbit of a discontinuity of T, let

$$w_{p,q}(x) = c_p, c_{p+1}, ..., c_{q-1}, c_q \text{ where } \tau(x) = ...c_{-1}, c_0, c_1, ...$$

This is a *block of length* $q-p$.

The map τ is not continuous as a map from $[0,1)$ with the standard topology to $\{1,2,...,d\}^{\mathbb{Z}}$ with the product topology. Observe that the left shift acts continuously on $\tau([0,1)) \subset \{1,2,...,d\}^{\mathbb{Z}}$. However, if the discontinuities of T have infinite and disjoint orbits (the Keane condition) then $\tau([0,1))$ is not closed in $\{1,2,...,d\}^{\mathbb{Z}}$ with the product topology. This is because the points immediately to the left of a discontinuity give finite blocks that do not converge to an infinite block. Let $\hat{X}$ be the closure of $\tau([0,1))$ in $\{1,2,...,d\}^{\mathbb{Z}}$ with the product topology. $\hat{X}$ results from to adding a countable number of points, the left hand sides of points in orbits of a discontinuity; $\hat{X}$ is a compact metric space. Let $f : \hat{X} \to [0,1)$ by $f|_{\tau([0,1))} = \tau^{-1}$ and extend f by continuity to the rest of $\hat{X}$. Notice that, unlike τ, the map f is continuous. Moreover the map is injective away from the orbit of discontinuities, where it is 2 to 1. The left shift S acts continuously on $\hat{X}$ and if T is not in the direction of a saddle connection then the action of S on $\hat{X}$ is measure conjugate to the action of T on $[0,1)$.

If x is in the orbit of a discontinuity let $w_{p,q}(x^+) = \lim_{y \to x^+} w_{p,q}(y)$. Let $w_{p,q}(x^-) = \lim_{y \to x^-} w_{p,q}(y)$. Let $\mathcal{B}_l(T) = \{a_1, ..., a_l : \bigcap_{i=1}^{l} T^{-i}(I_{a_i}) \neq \emptyset\}$. This is often called the set of allowed l blocks. Observe that the preimages under τ of allowed l blocks are bounded by discontinuities of T^l, 0, and 1. Note that $|\mathcal{B}_{l+1}(T)| \leq |\mathcal{B}_l(T)| + d - 1$ for all $l \geq 1$. That is all but $d-1$ l-blocks have a unique continuation to an $l+1$ block.

Assume that there exist half open intervals $J_1, ..., J_r$ and natural numbers $m_1, ..., m_r$ such that T^j is continuous (thus an isometry) on J_i for $0 \leq j \leq m_i$, $\bigcup_{i=1}^{r} \bigcup_{j=0}^{m_i} T^j(J_i) = [0,1)$ and $T^j(J_i) \cap T^{j'}(J_{i'}) = \emptyset$ when $0 \leq j < j' \leq m_i$, $0 \leq j' \leq m_{i'}$ and $j \neq j'$ if $i = i'$.We say $\bigcup_{j=1}^{m_i} T^j(J_i)$ are *Rokhlin towers*. $m_i + 1$ is called the *height* of the Rokhlin tower. Each $T^j(J_i)$ is called a *level*. Every word of $\tau([0,1)$ is a concatenation of $\omega_{0,m_i}(z_i)$ where $z_i \in J_i$. By construction, $y_i, z_i \in J_i$ implies that $\omega_{0,m_i}(z_i) = \omega_{0,m_i}(y_i)$. Also $\omega_{0,m_i-j}(T^j(y_i)) = \omega_{j,m_i}(y_i)$. In this way a set of Rokhlin towers at a fixed stage describes to a limited extent the dynamics of a system. As one takes Rokhlin towers with more and more levels one gains a better understanding of the dynamical system.

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